

INEQUALITIES AND GEOMETRY OF HYPERBOLIC-TYPE METRICS, RADIUS PROBLEMS AND NORM ESTIMATES

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THESIS CERTIFICATE

This is to certify that the thesis entitled **INEQUALITIES AND GEOMETRY OF HYPERBOLIC-TYPE METRICS, RADIUS PROBLEMS AND NORM ESTIMATES** submitted by **Swadesh Kumar Sahoo** to the Indian Institute of Technology Madras for the award of the degree of Doctor of Philosophy is a bonafide record of research work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Research Guide

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(S. Ponnusamy)

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ABSTRACT

KEYWORDS: The Apollonian, the Apollonian inner, the j , the λ -Apollonian, the Barbilian, Ferrand's, the inner, the K-P, and the quasihyperbolic metrics; John, quasi-isotropic, and uniform domains; the comparison property; isometry; analytic, convex, hypergeometric, starlike, strongly starlike, and univalent functions; coefficient inequality; pre-Schwarzian norm; integral transforms; and subordinations.

We consider certain inequalities among the Apollonian metric, the Apollonian inner metric, the j metric and the quasihyperbolic metric. We verify that whether these inequalities can occur in simply connected planar domains and in proper subdomains of \mathbb{R}^n ($n \geq 2$). We have seen from our verification that most of the cases cannot occur. This means that there are many restrictions on domains in which these inequalities can occur. We also consider two metrics j and d , and investigate whether a plane domain $D \subsetneq \mathbb{C}$, for which there exists a constant $c > 0$ with $j(z, w) \leq c d(z, w)$ for all $z, w \in D$, is a uniform domain. In particular, we study the case when d is the λ -Apollonian metric. We also investigate the question, whether simply connected quasi-isotropic domains are John disks and conversely. Isometries of the quasihyperbolic metric, the Ferrand metric and the K-P metric are also obtained in several specific domains in the complex plane.

In addition to the above, some problems on univalent functions theory are also solved. We denote by \mathcal{S} , the class of normalized univalent analytic functions defined in the unit disk. We consider some geometrically motivated subclasses, say \mathcal{F} , of \mathcal{S} . We obtain the largest disk $|z| < r$ for which $\frac{1}{r}f(rz) \in \mathcal{F}$ whenever $f \in \mathcal{S}$. We also obtain necessary and sufficient coefficient conditions for f to be in \mathcal{F} . Finally, we present the pre-Schwarzian norm estimates of functions from \mathcal{F} and that of certain convolution or integral transforms of functions from \mathcal{F} .

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ABBREVIATIONS

HC	Hyperbolic Center
HMA	Hyperbolic Medial Axis
K–P	Kulkarni–Pinkall
\mathcal{LU}	Locally Univalent
MA	Medial Axis

NOTATION

English Symbols

$\text{card}(A)$	cardinality of the set A
$\text{diam } A$	diameter of the set A
\mathcal{A}	class of normalized analytic functions in \mathbb{D}
$A \subset B$	A is a subset of B
$A \subsetneq B$	A is a proper subset of B
$B^n(x, r)$	Euclidean ball of center x with radius r
$B(x, r)$	disk with center x and radius r
B^n	Euclidean unit ball
B_z	extremal disk in a domain G centered at z
C^n	n -times continuously differentiable
\mathbb{C}	complex plane
\mathbb{D}	unit disk
$\tilde{d}(x, y)$	inner metric of the metric $d(x, y)$
$\{e_1, e_2, \dots, e_n\}$	standard basis of \mathbb{R}^n
$\ f\ $	pre-Schwarzian norm of f
$f * g$	Hadamard (convolution) product of f and g
$f \prec g$	f is subordinate to g
∂G	boundary of G
$\partial_r G$	set of rectifiably accessible points in ∂G
\overline{G}	closure of G
G^c	complement of G
\mathcal{H}	class of analytic functions in \mathbb{D}
\mathcal{H}_a	class of analytic functions in \mathbb{D} which take origin into $a \in \mathbb{C}$

H^n	upper half-space
h_D	hyperbolic metric defined on D
i_z	inversion in a circle centered at z with radius 1
j_G	j metric defined on G (it is used in two different meanings)
\mathcal{K}	class of convex functions
$\mathcal{K}(\alpha)$	class of convex functions of order α , $0 \leq \alpha \leq 1$
k_G	quasihyperbolic metric defined on G
\hat{K}_z	hyperbolic convex hull of the set $\partial B_z \cap \partial G$ in B_z
\mathbb{R}^n	Euclidean n -space
$\overline{\mathbb{R}^n}$	$\mathbb{R}^n \cup \{\infty\}$, the Möbius space
R_ζ	reciprocal of the curvature of ∂G at $\zeta \in \partial G$
$S^{n-1}(x, r)$	Euclidean sphere of center x with radius r
S^{n-1}	Euclidean unit sphere
\mathcal{S}	class of univalent functions
\mathcal{S}^*	class of starlike functions
$\mathcal{S}^*(\alpha)$	class of starlike functions of order α , $0 \leq \alpha \leq 1$
$\mathcal{SS}^*(\alpha)$	class of strongly starlike functions of order α , $0 < \alpha \leq 1$.
T_f	pre-Schwarzian derivative of f
xy	line through x and y
$[x, y]$	closed line segment between x and y
$(x, y]$ or $[x, y)$	half-open (or half-closed) segment between x and y
\widehat{xyz}	smallest angle between the vectors $x - y$ and $z - y$
x_i	i^{th} coordinate of $x \in \mathbb{R}^n$
$\operatorname{Re} z$	real part of z
$\operatorname{Im} z$	imaginary part of z

Greek Symbols

α_G	Apollonian metric defined on the domain G
$\tilde{\alpha}_G$	Apollonian inner metric defined on the domain G
α'_G	λ -Apollonian metric defined on G
$\bar{\alpha}_G(x; \theta)$	directed density of the Apollonian metric at x in the direction θ

$\delta(x)$	$d(x, \partial G)$, Euclidean distance of $x \in G$ to ∂G
$d(\gamma)$	d -length of γ
$\ell(\gamma)$	Euclidean length of γ
$\lambda_D(z_1, z_2)$	λ -length between z_1 and z_2
μ_D	K–P metric defined on D
σ_D	Ferrand’s metric defined on D

CHAPTER 1

INTRODUCTION

The thesis consists of six chapters. The purpose of this chapter is to give primitive motivations and backgrounds for the remaining chapters. In Section 1.1, we review brief introduction to the Apollonian metric which is a generalization of Poincaré's model of the hyperbolic metric with some geometric facts. In Section 1.2, we present some inequalities among certain hyperbolic-type metrics and their geometric characterizations in terms of domains. In Section 1.3, we present isometries of certain metrics with an aim to investigate the same behavior for other related metrics in specific domains. In Section 1.4, we give some motivations to study necessary and sufficient conditions for functions to be in some subclass of univalent functions in terms of Taylor's coefficient. In addition, we introduce the definition of the radius problem and collect some well-known results with a motivation to study for some subclasses of univalent analytic functions. Section 1.5 begins with the pre-Schwarzian norm of functions from certain well-known classes of locally univalent functions and ends with some related problems that we solved in last chapter. At last in Section 1.6, we summarize our investigations with some conclusion.

The thesis is organized with solutions to a number of problems. For example, we consider the following problems.

- Given some sets of inequalities among the Apollonian metric, the Apollonian inner metric [52], the j metric and the quasihyperbolic metric; we ask whether these can occur together in simply connected planar domains or in general domains of the Euclidean space \mathbb{R}^n ($n \geq 2$)!
- Can uniform domains be characterized in terms of inequalities between the j metric and the λ -Apollonian metric? What is the relationship between quasi-isotropic domains and John disks?
- What are the isometries of the quasihyperbolic metric, the Ferrand metric and the K-P metric?

- We identify some subclasses, say \mathcal{F} , of the class of normalized analytic univalent functions \mathcal{S} and find largest disk $|z| < r$ for which $\frac{1}{r}f(rz) \in \mathcal{F}$ whenever $f \in \mathcal{S}$. In addition, we find necessary and sufficient coefficient conditions for f to be in \mathcal{F} .
- Given some classes of univalent analytic functions, we obtain the pre-Schwarzian norm estimates of functions from the given classes as well as that of certain integral or convolution operators of functions from those classes.

In the thesis, we say non-empty open connected sets as domains.

HYPERBOLIC TYPE METRICS:

We begin with the definition of a metric as follows. A metric space is a non-empty set M together with a real valued function $d : M \times M \rightarrow \mathbb{R}$ (called a metric, or sometimes a distance function) such that for every $x, y, z \in M$ we have the following properties:

- $d(x, y) \geq 0$, with equality if and only if $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The Schwarz lemma, named after Hermann Amandus Schwarz, is a result in complex analysis about holomorphic functions defined on the unit disk. A variant of the Schwarz lemma can be stated that is invariant under analytic automorphisms on the unit disk, i.e. bijective holomorphic mappings of the unit disk to itself. This variant is known as the Schwarz–Pick lemma (after George Pick). The Schwarz–Pick lemma then essentially gives that a holomorphic map of the unit disk into itself decreases the distance of points in the Poincaré metric. In early 19th century, Poincaré used the unit ball and Lobachevsky used the half space as domains for their models. By the Riemann mapping theorem we know that any simply connected proper subdomain of the plane is conformally equivalent to the unit disk. So it is possible to define the hyperbolic metric in simply connected subdomains of the complex plane as well.

In contrast to the situation in the complex plane, the well-known hyperbolic metric is defined only in balls and half-spaces in \mathbb{R}^n when $n \geq 3$. Many researchers have proposed metrics that could take the place of the hyperbolic metric in analysis in higher dimensions.

Probably the most used one is the quasihyperbolic metric introduced by Gehring and Palka in [41]. This metric has the slight disadvantage is that it is not equal to the hyperbolic metric in a ball. Several metrics have also been proposed that are generalizations of the hyperbolic metric in the sense that they equal the hyperbolic metric if the domain of definition is a ball or a half-space. Some examples are the Apollonian metric [8], the Ferrand metric [30], the K–P metric [74] and Seittenranta’s metric [113]. Apart from the above metrics we also consider the j metric and the idea of inner metric in this thesis. Note that inner metric of the Apollonian metric is called the *Apollonian inner metric*, inner metric of the j metric is known as the quasihyperbolic metric and that of Seittenranta’s metric is the Ferrand metric. The common fact for all the above metrics is that they are defined in some proper subdomain of \mathbb{R}^n ($n \geq 2$) and are strongly affected by the geometry of the boundary of the domain. Because of this, we sometimes say these metrics as hyperbolic-type metrics. Most of the metrics described here have an invariance property in the sense of

$$(1.1) \quad d_D(x, y) = d_{f(D)}(f(x), f(y)),$$

for all $x, y \in D$ and for mappings f belonging to some fixed class, say the class of conformal maps, the class of Möbius maps and the class of similarities. Here $D \subsetneq \mathbb{R}^n$ is a metric space with the metric d .

In Chapter 4, we characterize f satisfying the relation (1.1) with respect to some of the hyperbolic-type path (or conformal path) metrics of the form

$$(1.2) \quad d_D(x, y) = \inf_{\gamma} \int_{\gamma} p(z) |dz|,$$

where $p(z)$ is a density function defined on D , $|dz|$ represents integration with respect to path-length, and the infimum is taken over all rectifiable paths γ joining $x, y \in D$. The *hyperbolic metric* $h_{\mathbb{D}}$ of the unit disk \mathbb{D} has the density function $2/(1 - |z|^2)$. In this case the infimum γ is attained for the non-euclidean segment from x to y , that is the arc of the circle through x and y orthogonal to the unit circle. The *hyperbolic metric* h_D of a simply connected plane domain D (other than \mathbb{C}) is obtained by transferring $h_{\mathbb{D}}$ to h_D by any conformal map of \mathbb{D} onto D . Indeed, if f maps \mathbb{D} onto D , then the *hyperbolic metric* of D is defined by

$$(1.3) \quad h_D(u, v) = h_{\mathbb{D}}(x, y) \quad \text{for } u = f(x), \ v = f(y) \text{ and } x, y \in \mathbb{D}.$$

See [9, 81, 82] and their references for basic properties of hyperbolic density.

Ferrand's metric [30] is defined by replacing the density function $p(z)$ in (1.2) with the function

$$(1.4) \quad \sigma_D(z) = \sup_{a,b \in \partial D} \frac{|a-b|}{|a-z||b-z|}.$$

The *K-P metric* [74] is defined by the density

$$(1.5) \quad \mu_D(x) = \inf \left\{ \lambda_B(x) : x \in B \subset D, B \text{ is a disk or a half-plane} \right\}.$$

Here λ_B is the density of the hyperbolic metric in B . Recall that if $B = B(x_0, r) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$, then

$$\lambda_B(x) = \frac{2r}{r^2 - |x - x_0|^2}.$$

1.1. The Apollonian Metric

The Apollonian metric was first introduced by Barbilian [4] in 1934–35 and then rediscovered by Beardon [8] in 1998. This metric has also been considered in [17, 39, 107, 113] and in [49, 50, 51, 52, 53, 57, 56, 64, 65, 66]. It should also be noted that the same metric has been studied from a different perspective under the name of the Barbilian metric for instance in [4, 5, 6, 16, 18, 68], cf. [19] for a historical overview and more references. One interesting historical point, made in [19, 20], is that Barbilian himself proposed the name “Apollonian metric” in 1959, which was later independently coined by Beardon [8]. More recently, the Apollonian metric has especially been studied by Hästö and Ibragimov in a series of articles, see e.g. [49]–[53], [57, 56] and [64]–[66]. An interesting fact is that the Apollonian metric is also studied with certain group structures [84].

We denote by $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ the one point compactification of \mathbb{R}^n . The *Apollonian metric* is defined for $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ by

$$\alpha_G(x, y) := \sup_{a,b \in \partial G} \log \frac{|a-y||b-x|}{|a-x||b-y|}$$

(with the understanding that $|\infty - x|/|\infty - y| = 1$). It is in fact a metric if and only if the complement of G is not contained in a hyperplane and only a pseudometric otherwise, as

was noted in [8, Theorem 1.1]. Some of the main reasons for the interest in the metric are that

1. the formula has a very nice geometric interpretation (see Subsection 1.1.1);
2. it is invariant under Möbius map;
3. it equals the hyperbolic metric in balls and half-spaces;
4. it is monotone: $\alpha_{G_1}(x, y) \leq \alpha_{G_2}(x, y)$ whenever $x, y \in G_2 \subset G_1$; and
5. it is complete: $\alpha_G(x_n, y) \rightarrow \infty$ as $x_n \rightarrow \partial G$ for each $y \in G$.

We next define the Apollonian metric in a different approach called the *Apollonian balls approach*. This is the reason we say the metric as *Apollonian metric*. The name of the balls are called the Apollonian balls, because they satisfy the definition of Apollonian circles (see Apollonian circles theorem in [22]).

1.1.1. The Apollonian balls approach

In this subsection we present the Apollonian balls approach which gives a geometric interpretation of the Apollonian metric.

A map $f : \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$ defined by

$$f(x) = a + \frac{r^2(x - a)}{|x - a|^2}, \quad f(\infty) = a, \quad f(a) = \infty$$

is called an *inversion* in the sphere

$$S^{n-1}(a, r) := \{z \in \mathbb{R}^n : |z - a| = r\}$$

for $x, a \in \mathbb{R}^n$ and $r > 0$. For $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ we define the following [113]

$$q_x := \sup_{a \in \partial G} \frac{|a - y|}{|a - x|}, \quad q_y := \sup_{b \in \partial G} \frac{|b - x|}{|b - y|}.$$

The numbers q_x and q_y are called the Apollonian parameters of x and y (with respect to G) and by definition $\alpha_G(x, y) = \log(q_x q_y)$. This gives an equivalent form of the Apollonian metric. The balls (in $\overline{\mathbb{R}^n}$!)

$$B_x := \left\{ z \in \overline{\mathbb{R}^n} : \frac{|z - x|}{|z - y|} < \frac{1}{q_x} \right\} \quad \text{and} \quad B_y := \left\{ w \in \overline{\mathbb{R}^n} : \frac{|w - y|}{|w - x|} < \frac{1}{q_y} \right\},$$

are called the *Apollonian balls* about x and y , respectively. Note that these balls are nothing but the Euclidean balls, see Item 4 below. We collect some immediate results regarding these balls; similar results obviously hold with x and y interchanged.

1. We have $x \in B_x \subset G$ and $\overline{B_x} \cap \partial G \neq \emptyset$.
2. If $\infty \notin G$, we have $q_x \geq 1$. If, moreover, $\infty \notin \overline{G}$, then $q_x > 1$.
3. We have $B_x \cap B_y = \emptyset$. If G is bounded then $\partial B_x \cap \partial B_y = \emptyset$. In other words, if ∂B_x intersects ∂B_y then $B_x \cup B_y = G$ (in fact, $\partial B_x = \partial B_y = \partial G$).
4. If $q_x > 1$, x_0 denotes the center of B_x and r_x its radius, then

$$x_0 = x + \frac{x - y}{q_x^2 - 1} \quad \text{and} \quad r_x = \frac{q_x |x - y|}{q_x^2 - 1};$$

and hence

$$|x - x_0| = \frac{|x - y|}{q_x^2 - 1} = \frac{r_x}{q_x}.$$

5. If i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y respectively, then $y = i_x(x) = i_y(x)$.
6. We have $q_x - 1 \leq |x - y|/\delta(x) \leq q_x + 1$.

1.2. Inequalities and Geometry

As a motivation for the study of inequalities and geometry, we mention that many inequalities among hyperbolic-type metrics have been previously studied by well-known authors and some have geometrical characterizations. For example, quasidisks and uniform domains characterizations are well established in terms of inequalities among hyperbolic-type metrics.

First we bring out the inequalities between the Apollonian metric α_G and the hyperbolic metric h_G in simply connected plane domain G . In this case, the Apollonian metric α_G satisfies the inequality $\alpha_G \leq 2h_G$ (see [8, Theorem 1.2]). Furthermore, it is shown in [8, Theorem 6.1] that for bounded convex plane domains the Apollonian metric satisfies $h_G \leq 4 \sinh [\frac{1}{2}\alpha_G]$, and by considering the example of the infinite strip $\{x + iy : |y| < 1\}$, that the best possible constant in this inequality is at least π . Later in 1997, Rhodes [107] improved the previous concept to general convex plane domain which says that

$h_G < 3.627 \sinh[\frac{1}{2}\alpha_G]$ and the best possible constant is at least 3.164. Also due to Bear-don [8], any bounded plane domain satisfying $\alpha_G \leq h_G$ is convex. On the other hand, Gehring and Hag [39] established that any domain $G \subsetneq \mathbb{C}$ of hyperbolic type is a disk if and only if $\alpha_G \leq h_G$, where disk means the image of unit disk under a Möbius map. There are characterization for quasidisks in terms of the Apollonian metric and the hyperbolic metric as well. A *quasidisk* is the image of a disk under a quasiconformal self map of $\overline{\mathbb{R}^2}$. Note that, it is always a non-trivial task to obtain a specific quasiconformal map when a domain is quasidisk. For some concrete examples of quasidisks and corresponding quasiconformal mappings see [42] (see also [35]). However, several characterizations [35] of quasidisks have been obtained from which it became convenient to say whether a domain is quasidisk. For example, it is well-known that simply connected uniform domains are quasidisks and it is not always difficult to present a proof for a simply connected domain to be uniform. There are also characterizations of quasidisks in terms of hyperbolic-type metrics. Due to Gehring and Hag [39], a simply connected domain $G \subsetneq \overline{\mathbb{R}^2}$ is a quasidisk if and only if there is a constant c such that $h_G(z_1, z_2) \leq c \alpha_G(z_1, z_2)$ for $z_1, z_2 \in G$. For several other interesting characterizations of quasidisks, see [35].

Let $\gamma: [0, 1] \rightarrow G \subset \mathbb{R}^n$ be a path. If d is a metric in G , then the d -length of γ is defined by

$$d(\gamma) := \sup \sum_{i=0}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is taken over all k ($< \infty$) and all sequences $\{t_i\}$ satisfying $0 = t_0 < t_1 < \dots < t_k = 1$. All paths in the thesis are assumed to be rectifiable, that is, to have finite Euclidean length. The *inner metric* of d is defined by

$$\tilde{d}(x, y) := \inf_{\gamma} d(\gamma),$$

where the infimum is taken over all paths γ connecting x and y in G . By repeated use of the triangle inequality it follows that $d \leq \tilde{d}$. We denote the inner metric of the Apollonian metric by $\tilde{\alpha}_G$ and call it the *Apollonian inner metric*. Since α_G is a metric except in a few domains, it is not reasonable to expect that $\tilde{\alpha}_G$ is a metric in every domain. In fact, $\tilde{\alpha}_G$ is a metric if and only if the complement of G is not contained in an $(n - 2)$ -dimensional hyperplane in \mathbb{R}^n [52, Theorem 1.2]. In the same paper Hästö established an explicit

integral formula for the Apollonian inner metric [52, Theorem 1.4], and proved that for most domains there exists a geodesic connecting two arbitrary points [52, Theorem 1.5].

We now define the other two metrics that we consider in Chapter 2. Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$.

The metric j_G is defined by

$$(1.6) \quad j_G(x, y) := \log \left(1 + \frac{|x - y|}{\min\{\text{dist}(x, \partial G), \text{dist}(y, \partial G)\}} \right),$$

see [119]. In a slightly different form of this metric was defined in [40]. Indeed, the j metric from [40] is defined by

$$(1.7) \quad j_G(x, y) = \log \left(1 + \frac{|x - y|}{\text{dist}(x, \partial G)} \right) \left(1 + \frac{|x - y|}{\text{dist}(y, \partial G)} \right) \quad \text{for all } x, y \in G$$

(see also Chapter 3 of the present thesis for this definition in plane domains). Note that both the metrics defined by (1.6) and (1.7) are equivalent. In further discussion of the current chapter and in Chapter 2 we use the notation j defined by (1.6); and in Chapter 3 we use the notation j defined by (1.7). The metric j_G is complete, monotone and invariant under similarity maps (see e.g. [36]). It is not Möbius invariant, but a Möbius quasi-invariant [40, Theorem 4].

The *quasihyperbolic metric* from [41] is defined by

$$k_G(x, y) := \inf_{\gamma} \int_{\gamma} \frac{|dz|}{\text{dist}(z, \partial G)},$$

where the infimum is taken over all paths γ joining x and y in G . We recall from [40] that the quasihyperbolic geodesic segment exists between each pair of points $x, y \in G$, i.e. the length minimizing curve γ joining x and y for $k_G(x, y)$ exists. The metric k_G is complete, monotone and changes at most by the factor 2 under a Möbius map (i.e. it is Möbius quasi-invariant, see [36, 40, 41]). Note that the quasihyperbolic metric is the inner metric of the j_G metric, see for instance [49, Lemma 5.3], and hence we have the fundamental inequality

$$j_G(x, y) \leq k_G(x, y) \quad \text{for all } x, y \in G$$

which is used several times in the present thesis.

In order to describe further discussion on inequalities and geometry we define some relations on the set of metrics.

Definition 1.1. Let d and d' be metrics on G .

1. We write $d \lesssim d'$ if there exists a constant $K > 0$ such that $d \leq Kd'$. Similarly for the relation $d \gtrsim d'$.
2. We write $d \approx d'$ if $d \lesssim d'$ and $d \gtrsim d'$.
3. We write $d \ll d'$ if $d \lesssim d'$ and $d \not\gtrsim d'$.
4. We write $d \leqslant d'$ if $d \not\lesssim d'$ and $d \not\gtrsim d'$.

Recall that $\alpha_G \leq 2j_G$ in every domain $G \subsetneq \mathbb{R}^n$ by [8, Theorem 3.2]. Also, it was shown in [113, Theorem 4.2] that if $G \subsetneq \mathbb{R}^n$ is convex, then $j_G \leq \alpha_G$. So $\alpha_G \approx j_G$ in convex domains. The condition $\alpha_G \approx j_G$ is also connected to various interesting properties, see for example [53, Theorem 1.3]. In the paper [53] the term *comparison property* was introduced for the relation $\alpha_G \approx j_G$. In [53], Hästö has given a geometrical characterization, in terms of an interior double ball condition, of those domains satisfying the comparison property. Additionally, the inequalities $\tilde{\alpha}_G \approx k_G$, $\alpha_G \approx \tilde{\alpha}_G$ and $\alpha_G \approx k_G$, which have been called quasi-isotropy, Apollonian quasiconvexity and A -uniformity, respectively, have some nice geometric interpretations and have been considered in [49, 50, 51].

We now recall the definition of uniform domains introduced by Martio and Sarvas in [80, 2.12] (see also [40, (1.1)] and Definition 2.2 in Chapter 2 for equivalent formulations).

Definition 1.2. A domain G is called a *uniform domain* provided there exists a constant c with the property that each pair of points $z_1, z_2 \in G$ can be joined by a path $\gamma \subset G$ satisfying

$$\ell(\gamma) \leq c|z_1 - z_2| \quad \text{and} \quad \min_{j=1,2} \ell(\gamma[z_j, z]) \leq c \operatorname{dist}(z, \partial G) \quad \text{for all } z \in \gamma.$$

Here $\gamma[z_j, z]$ denotes the part of γ between z_j and z .

Here we remark that the first condition is called the quasiconvexity condition and the second one is called the double cone (John) condition. More precisely, if we remove

the first condition from Definition 1.2, then we call the domain as *John domain*. Simply connected John domains are called the *John disks* [83]. Similarly, if we omit the second condition from the definition the property would be meant for the quasiconvex domain. Thus we conclude that every uniform domain is John domain as well as quasiconvex domain.

Since the quasihyperbolic metric k_G is the inner metric of the metric j_G , we have $j_G \leq k_G$ for any domain $G \subsetneq \mathbb{R}^n$. On the other hand, due to Gehring and Osgood (see [40, Corollary 1]), a domain $G \subsetneq \mathbb{R}^n$ is uniform if and only if $k_G \lesssim j_G$ holds. Although the inequality in [40] was stated in the form $k_G \leq c j_G + d$, it has been proved later that these two forms are equivalent (see for instance [38, 49] and [119, 2.50(2)]). This condition is also equivalent to $\tilde{\alpha}_G \lesssim j_G$, see [63, Theorem 1.2]. Thus we have a geometric characterization of domains satisfying these inequalities as well.

The above observations motivate us to study certain inequalities among the Apollonian metric α_G , the Apollonian inner metric $\tilde{\alpha}_G$, the j_G metric and the quasihyperbolic metric k_G ; and their geometric meaning which helps us to form Chapter 2 in this thesis.

The brief idea of Chapter 2 is as follows: Let us first of all note that the following inequalities hold in every domain $G \subsetneq \mathbb{R}^n$:

$$(1.8) \quad \alpha_G \lesssim j_G \lesssim k_G \quad \text{and} \quad \alpha_G \lesssim \tilde{\alpha}_G \lesssim k_G.$$

The first two are from [8, Theorem 3.2] and the second two from [49, Remark 5.2 and Corollary 5.4]. We see that of the four metrics to be considered, the Apollonian is the smallest and the quasihyperbolic is the largest. We will undertake a systematic study of which of the inequalities in (1.8) can hold in the strong form with \ll and which of the relations $j_G \ll \tilde{\alpha}_G$, $j_G \approx \tilde{\alpha}_G$ and $j_G \gg \tilde{\alpha}_G$ can hold.

1.3. Isometries of Hyperbolic-type Metrics

If a metric is of interest, then so are its isometries. By the *isometry problem* for the metric d we mean characterizing mappings $f : D \rightarrow \mathbb{R}^2$ which satisfy (1.1) for all $x, y \in D$. Although not a part of our result but as a motivation; before going to start on

path metrics, we give an overview on isometries of the Apollonian metric. Isometries of the j metric were studied by Hästö, Ibragimov and Lindén [55, 59].

We recall that the Apollonian metric was introduced by Beardon in 1998. However, it remained an open question that what are all its isometries. Beardon first raised this question and studied whether the Apollonian isometries are only Möbius maps. He proved that conformal mappings of plane domains, whose boundary is a compact subset of the extended negative real axis that contains at least three points, which are Apollonian isometries are indeed Möbius mappings, [8, Theorem 1.3]. In [57], Hästö and Ibragimov have established that this is true in the case of all open sets with regular boundary. On the other hand, in [56], the same authors have found Apollonian isometries of plane domains but without assumption on regularity of the boundary. They proved that Möbius mappings in plane domain are the isometries of the Apollonian metric as long as the domain has at least three boundary points. We note that the last two observations on isometries of the Apollonian metric are very recent ones. However, a few years ago, Gehring and Hag [39] have considered only the sense preserving Apollonian isometries in disks and showed that they are always the restriction of Möbius maps. Independently, in 2003, Hästö has generalized the above idea of Gehring and Hag (see [49, 50]) to \mathbb{R}^n as well.

Next we keep an eye on the study of isometries of hyperbolic-type conformal metrics. There are three steps in characterizing isometries of a conformal metric by showing that they are

- (1) conformal; (2) Möbius; (3) similarities.

The step (1) has been carried out by Martin and Osgood [79, Theorem 2.6] for arbitrary domains assuming only that the density is continuous, so there is no more work to do there. Note that step (2) is trivial in dimensions 3 and higher (because all conformal maps are nothing but Möbius maps), and that step (3) is not relevant for Möbius invariant metrics like the K–P metric and Ferrand’s metric.

Among the conformal metrics we mainly concentrate on the quasihyperbolic metric, the K–P metric and the Ferrand metric. The work of Hästö [54] on steps (2) and (3) is very recent one. He proved that, except for the trivial case of a half-plane, the quasihyperbolic

isometries are similarities with some assumptions of smoothness on the domain boundary. Indeed, for example, he showed that a quasihyperbolic isometry which is also a Möbius transformation is a similarity provided the domain is a C^1 domain which is not a half-plane (see [54, Proposition 2.2]). Here C^k domain means its boundary is locally the graph of a C^k function. From [79, Theorem 2.8] we know that every quasihyperbolic isometry is conformal. In dimensions three and higher all conformal mappings are Möbius. So certainly, Hästö could be able to generalize his above result to higher dimensions as well. He has also generalized the results to C^3 domains. Regarding step (2), the reader is referred to [54, Section 4]. The work by Herron, Ibragimov and Minda [60] shows that all isometries of the K–P metric are Möbius mappings except in simply and doubly connected domains.

The above ideas of isometries encourage us to study isometries in other specific domains and for the Ferrand metric as well. This leads to form a survey article along with some new results in Chapter 4.

UNIVALENT FUNCTIONS THEORY:

Univalent function theory is a classical area in the branch of complex analysis. We know that a *function* is a rule of correspondence between two sets such that there is a unique element in the second set assigned to each element in the first set. A function on a domain is called *univalent* if it is one-to-one. For example, any function $\phi_a(z)$ of the unit disk to itself, defined by

$$(1.9) \quad \phi_a(z) = \frac{z + a}{1 + \bar{a}z}$$

is univalent, where $|a| < 1$. Various other terms are used for this concept, e.g. *simple*, *schlicht* (the German word for simple). We are mainly interested in univalent functions that are also analytic in the given domain. We refer the standard books by Duren [29], Goodman [44] and Pommerenke [95] for this theory.

The theory of univalent functions is so vast and complicated so that certain simplifying assumptions are necessary. If $g(z)$ is analytic in the unit disk \mathbb{D} , it has the Taylor series

expansion

$$g(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{n=0}^{\infty} b_n z^n.$$

We observe that if $g(z)$ is univalent in \mathbb{D} then the function $f(z) = (g(z) - b_0)/b_1$ is also univalent in \mathbb{D} and conversely. Setting $b_n/b_1 = a_n$ in the above expansion of g we arrive at the normalized form

$$(1.10) \quad f(z) = z + a_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n.$$

Here we note that, the above normalized form of the function f satisfies the relation $f(0) = 0 = f'(0) - 1$. The well-known example in this class is *the Koebe function*, $k(z)$, defined by

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n$$

which is an extremal function for many subclasses of the class of univalent functions.

It is natural to ask the following two questions about the representation (1.10).

- (1) Given the sequence of coefficients $\{a_n\}$, how does it influence some geometric properties of $f(z)$?
- (2) Given some properties of $f(z)$, how does this property affect the coefficients in (1.10)?

We denote by \mathcal{A} , the class of analytic functions f in \mathbb{D} of the form (1.10) and \mathcal{S} denotes the class of all functions $f \in \mathcal{A}$ that are univalent in \mathbb{D} . Functions in the class \mathcal{S} have a nice geometric property that the range of the function contains a disk of radius at most $1/4$, because an extremal function $k(z)$ maps the unit disk onto the whole plane except a slit along the negative real axis from $-1/4$ to ∞ . This result is known as Koebe's one-quarter theorem.

Many authors have studied a number of subclasses of univalent functions as well. Among those, the class of convex and starlike functions are the most popular and interesting because of their simple geometric properties.

A domain $D \subset \mathbb{C}$ is said to be *starlike* with respect to a point $z_0 \in D$ if the line segment joining z_0 to every other point $z \in D$ lies entirely in D . A function $f \in \mathcal{S}$ is said

to be a *starlike function* if $f(\mathbb{D})$ is a domain starlike with respect to origin. The class of all starlike functions is denoted by \mathcal{S}^* . A typical example of a function in this class is the Koebe function and as an extremal function, the range of every function $f \in \mathcal{S}^*$ contains the disk $|w| < \frac{1}{4}$.

A domain $D \subset \mathbb{C}$ is said to be *convex* if it is starlike with respect to each of its points; that is, if the line segment joining any two points of D lies completely in D . Similar to starlike functions a function $f \in \mathcal{S}$ is said to be convex if $f(\mathbb{D})$ is a convex domain. The class of all convex functions is denoted by \mathcal{K} . The function

$$\ell(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

belongs to the class \mathcal{K} and maps \mathbb{D} onto the half-plane $\operatorname{Re}\{w\} > -\frac{1}{2}$. This function plays a role of extremal function for many problems in the class \mathcal{K} as well. The range of every function $f \in \mathcal{K}$ contains the disk $|w| < \frac{1}{2}$. We note that $z\ell'(z) = k(z)$.

An analytic description of starlike functions is that

$$f \in \mathcal{S}^* \text{ if and only if } \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \text{ for } z \in \mathbb{D}.$$

Convex functions have also a similar description:

$$f \in \mathcal{K} \text{ if and only if } \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \text{ for } z \in \mathbb{D}.$$

The two preceding descriptions reveal an interesting close analytic characterization between convex and starlike functions. This says that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. This was first observed by Alexander [2] in 1915 and then onwards the result is known as *Alexander's theorem*. In view of this, the one-to-one correspondence between \mathcal{K} and \mathcal{S}^* is given by the well-known *Alexander transform* defined by

$$(1.11) \quad J[f](z) = \int_0^z \frac{f(t)}{t} dt.$$

That is, $J[f]$ is convex if and only if f is starlike. Here we remark that the Alexander transform (1.11), in general, does not take an univalent function into another univalent function (see [29, Theorem 8.11]). The above properties of Alexander's transform motivate to study several other generalized transforms in the theory of univalent functions.

1.4. Coefficient Conditions and Radii Problems

We begin with the following conjecture [13, 14]:

Bieberbach's Conjecture. If $f \in \mathcal{S}$, then for each $n \geq 2$ we have $|a_n| \leq n$.

The conjecture was unsolved for about 70 years although it had been proved in several special cases $n = 2, 3, 4, 5, 6$ [95, page 24] and many other subclasses of \mathcal{S} . But finally, Louis de Branges [21] settled it in the affirmative in 1985. Several other type of coefficient estimates, such as sufficient conditions for f to be in several subclasses of \mathcal{S} , are also well-established. For example (see Goodman [43]) if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies $\sum_{n=2}^{\infty} n|a_n| \leq 1$ then $f \in \mathcal{S}^*$.

The problem of estimating the radius of various classes of univalent functions has attracted a certain number of mathematicians involved in geometric function theory. For a systematic survey of radius problems, we refer to [44, Chapter 13].

Let \mathcal{F} and \mathcal{G} be two subclasses of \mathcal{A} . If for every $f \in \mathcal{F}$, $r^{-1}f(rz) \in \mathcal{G}$ for $r \leq r_0$, and r_0 is the largest number for which this holds, then we say that r_0 is the \mathcal{G} radius (or the radius of the property connected to \mathcal{G}) in \mathcal{F} . This implies that if $r > r_0$, there is at least one $f \in \mathcal{F}$ such that $r^{-1}f(rz) \notin \mathcal{G}$. Here our main aim is to obtain r_0 . There are many results of this type that have been studied in the theory of univalent functions. For example, the radius of convexity for the class \mathcal{S} is known to be $2 - \sqrt{3}$ and that of starlikeness for the same class is $\tanh \pi/4 \approx 0.656$.

As a motivation of the discussion in Section 1.4, we form Chapter 5.

An outline of Chapter 5 is as follows: We define a subclass $\mathcal{S}_p(\alpha)$, $-1 \leq \alpha \leq 1$, of starlike functions in the following way [108]:

$$\mathcal{S}_p(\alpha) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)} - \alpha, \quad z \in \mathbb{D} \right\}.$$

Geometrically, $f \in \mathcal{S}_p(\alpha)$ if and only if the domain values of $zf'(z)/f(z)$, $z \in \mathbb{D}$, is the parabolic region $(\operatorname{Im} w)^2 \leq (1 - \alpha)[2\operatorname{Re} w - (1 + \alpha)]$. We determine necessary and sufficient coefficient conditions for certain class of functions to be in $\mathcal{S}_p(\alpha)$. Also, radius properties are considered for $\mathcal{S}_p(\alpha)$ -class in the class \mathcal{S} . We consider another subclass of the class of

univalent functions which has recent interest as follows. A function $f \in \mathcal{A}$ is said to be in $\mathcal{U}(\lambda, \mu)$ if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| \leq \lambda \quad (|z| < 1)$$

for some $\lambda \geq 0$ and $\mu > -1$. We find disks $|z| < r := r(\lambda, \mu)$ for which $\frac{1}{r}f(rz) \in \mathcal{U}(\lambda, \mu)$ whenever $f \in \mathcal{S}$. In addition to a number of new results, we also present several new sufficient conditions for f to be in the class $\mathcal{U}(\lambda, \mu)$.

1.5. Pre-Schwarzian Norm

We recall that the class \mathcal{S} is preserved under disk automorphism (also called the *Koebe transform*). More precisely this means that if $f \in \mathcal{S}$ and

$$g(z) = \frac{f(\phi_a(z)) - f(a)}{(1 - |a|^2)f'(a)} = z + \left(\frac{1}{2}(1 - |a|^2) \frac{f''(a)}{f'(a)} - \bar{a} \right) z^2 + \cdots,$$

then g is also in the class \mathcal{S} , where $\phi_a(z)$ is defined by (1.9). The derivative quantity $T_f := f''/f'$ is called the *pre-Schwarzian derivative* of f or *logarithmic derivative* of f' . By Bieberbach's theorem and an easy simplification, we obtain

$$\left| (1 - |a|^2) \frac{f''(a)}{f'(a)} - 2\bar{a} \right| \leq 4,$$

where equality holds for a suitable rotation of the Koebe function. Consequently, one has

$$\sup_{|z| < 1} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| \leq 6$$

for $f \in \mathcal{S}$. This can also be seen from the result of Martio and Sarvas [80, Item 4.6] and Osgood [93, Lemma 1], which says about an upper bound property of the pre-Schwarzian derivative in terms of the quasihyperbolic density in a proper subdomain of the complex plane. The inequality is sharp, that is, we cannot replace the constant 6 by a smaller number and it can be seen by considering the Koebe function. This motivates us to study the quantity

$$(1.12) \quad \|f\| = \sup_{|z| < 1} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right|,$$

in the theory of univalent functions. We usually say this quantity as *pre-Schwarzian norm* of the function f . In general, the assumption on f can be restricted to *locally univalent* functions, namely, $\mathcal{LU} := \{f \in \mathcal{A} : f'(z) \neq 0, z \in \mathbb{D}\}$. We may regard \mathcal{LU} as

a vector space over \mathbb{C} not in the usual sense, but in the sense of Hornich operations (see [62, 70, 123]) defined by

$$(f \oplus g)(z) = \int_0^z f'(w)g'(w) dw \quad \text{and} \quad (\alpha \star f)(z) = \int_0^z \{f'(w)\}^\alpha dw$$

for $f, g \in \mathcal{LU}$ and $\alpha \in \mathbb{C}$, where the branch of $(f')^\alpha = \exp(\alpha \log f')$ is taken so that $(f')^\alpha(0) = 1$.

The pre-Schwarzian norm has significance in the theory of Teichmüller spaces (see e.g. [3]) as well. We remark that the norm $\|f\|$ is nothing but the Bloch semi-norm of the function $\log f'$ (see, for example, [95]). We have before already seen that $\|f\| \leq 6$ if f is univalent in \mathbb{D} , and it is well-known that if $\|f\| \leq 1$ then f is univalent in \mathbb{D} , and these bounds are sharp (see [10, 11]). Furthermore, $\|f\| < \infty$ if and only if f is uniformly locally univalent; that is, there exists a constant $\rho = \rho(f)$, $0 < \rho \leq 1$, such that f is univalent in each disk of hyperbolic radius $\tanh^{-1} \rho$ in \mathbb{D} , i.e. in each Apollonius (or Apollonian) disk

$$\left\{ z \in \mathbb{C} : \left| \frac{z-a}{1-\bar{a}z} \right| < \rho \right\}, \quad |a| < 1$$

(see [123, 124]). Note that the above disk is called the Apollonian disk, because it has the same nature as in the Apollonian balls defined in Subsection 1.1.1 with $q_a = 1/(\rho|a|)$. Here we observe from Property 5 of Subsection 1.1.1 that, the well-known inversion relation, a and $1/\bar{a}$ are the inverse points with respect to the unit circle. The set of all f with $\|f\| < \infty$ is a nonseparable Banach space (see [123, Theorem 1]). For more geometric and analytic properties of f relating the norm, see [72]. Many authors have given norm estimates for classical subclasses of univalent functions (see for example [27, 73, 92, 117, 125]).

For $f \in \mathcal{S}$, although its Alexander transform $J[f]$ is not in \mathcal{S} , it is locally univalent and so it is reasonable to obtain the norm estimates for the Alexander transform of certain classes of analytic functions. For example, it has been obtained in [70] that $\|J[f]\| \leq 4$ for $f \in \mathcal{S}$ and the inequality is sharp.

A simple generalization of \mathcal{S}^* is the so-called class of all *starlike functions of order* α , $0 \leq \alpha \leq 1$, denoted by $\mathcal{S}^*(\alpha)$. Indeed, $f \in \mathcal{S}^*(\alpha)$ if and only if $\operatorname{Re}(zf'(z)/f(z)) \geq \alpha$ in \mathbb{D} . Here we remark that the later inequality is strict except for $\alpha = 1$. We set $\mathcal{S}^*(0) = \mathcal{S}^*$.

Similarly, a function $f \in \mathcal{S}$ is said to be *convex of order α* if $\operatorname{Re}(1 + zf''(z)/f'(z)) \geq \alpha$. This class is denoted by $\mathcal{K}(\alpha)$. Like in the starlikeness we set $\mathcal{K}(0) = \mathcal{K}$.

In 1999, Yamashita [125] proved that if $f \in \mathcal{S}^*(\alpha)$ then $\|f\| \leq 6 - 4\alpha$ and $\|J[f]\| \leq 4(1 - \alpha)$ (or equivalently, $\|f\| \leq 4(1 - \alpha)$ for f convex of order α) for $0 \leq \alpha < 1$. Both the inequalities are sharp (see also [27, Theorem A]). There are many classes of functions f for which the norm $\|f\|$ is finite. We remark that if f is bounded, it may happen that $\|f\| = \infty$. For instance, the function

$$z \mapsto f(z) = \exp \frac{z+1}{z-1}$$

in the unit disk shows that $T_f(z) = -2z/(1-z)^2$ and hence, $\|f\| \rightarrow \infty$ as $z \rightarrow 1^-$.

Let us denote \mathcal{H} for the class of functions f analytic in the unit disk \mathbb{D} and \mathcal{H}_a will denote the subclass $\{f \in \mathcal{H} : f(0) = a\}$, for $a \in \mathbb{C}$. We say that a function $\varphi \in \mathcal{H}$ is *subordinate* to $\psi \in \mathcal{H}$ and write $\varphi \prec \psi$ or $\varphi(z) \prec \psi(z)$ if there is a Schwarz function ω (i.e. a function $\omega \in \mathcal{H}_0$ with $|\omega| < 1$ in \mathbb{D}) satisfying $\varphi = \psi \circ \omega$ in \mathbb{D} . Note that the condition $\varphi \prec \psi$ is equivalent to the conditions $\varphi(\mathbb{D}) \subset \psi(\mathbb{D})$ and $\varphi(0) = \psi(0)$ when ψ is univalent.

If $f, g \in \mathcal{H}$, with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then the *Hadamard product* (or *convolution*) of f and g is defined by the function

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

As a motivation in Chapter 6, we consider the class

$$\mathcal{K}(A, B) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D} \right\},$$

where $-1 \leq B < A \leq 1$ and \prec denotes the subordination. For $0 < b \leq c$, define $B_{b,c}[f]$ by

$$B_{b,c}[f](z) = zF(1, b; c; z) * f(z),$$

where $F(a, b; c; z)$ is the Gauss *hypergeometric function* defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad z \in \mathbb{D},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol (here $(a)_0 = 1$) and c is not a non-positive integer. We have the well-known derivative formula

$$F'(a, b; c; z) = \frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a+1, b+1; c+1; z).$$

As a special case of the Euler integral representation for the hypergeometric function, one has

$$F(1, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{1}{1-tz} t^{b-1} (1-t)^{c-b-1} dt, \quad z \in \mathbb{D}, \quad \operatorname{Re} c > \operatorname{Re} b > 0.$$

Using this representation we have, for $f \in \mathcal{A}$, the convolution transform

$$zF(1, b; c; z) * f(z) = z \left(F(1, b; c; z) * \frac{f(z)}{z} \right).$$

Therefore, we obtain the integral convolution which defines the (hypergeometric) operator $B_{b,c}[f]$ in the following form

$$B_{b,c}[f](z) := zF(1, b; c; z) * f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \frac{f(tz)}{t} dt$$

so that

$$(B_{b,c}[f])'(z) = F(1, b; c; z) * f'(z).$$

We obtain sharp pre-Schwarzian norm estimates for functions in $\mathcal{K}(A, B)$. In addition, we also present sharp norm estimates for $B_{b,c}[f](z)$ when f ranges over the class $\mathcal{K}(A, B)$.

Some particular cases need special attention. For example, if $c = b+1$ and $b = \gamma+1$, then one has the well-known *Bernardi transform* $B_\gamma[f] := B_{\gamma+1, \gamma+2}[f]$ defined by

$$(1.13) \quad B_\gamma[f](z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt = zF(1, \gamma+1; \gamma+2; z) * f(z),$$

for $\gamma > -1$. We observe that $B_0[f] = J[f]$ and $B_1[f] = L[f]$, where $J[f]$ and $L[f]$ are respectively the Alexander transform of f , and the Libera transform of f .

Also, similar norm estimates have been established for the class

$$\mathcal{F}_\beta = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}\beta, \quad z \in \mathbb{D} \right\},$$

where $\frac{2}{3} < \beta \leq 1$. The class \mathcal{F}_β and its special case $\mathcal{F}_1 = \mathcal{F}$ have been studied, for example, in [97, 99, 102] but for different purposes. In [97, Eq. (16)] it has been shown that if $f \in \mathcal{F}$, then one has

$$\left| \frac{zf'(z)}{f(z)} - \frac{2}{3} \right| < \frac{2}{3}, \quad z \in \mathbb{D}; \quad \text{i.e.} \quad \frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z}, \quad z \in \mathbb{D}.$$

Thus, $\mathcal{F}_\beta \subset \mathcal{F} \subset \mathcal{S}^*$ for $\frac{2}{3} < \beta \leq 1$. Note that each $f \in \mathcal{S}^*$ has the well-known analytic characterization:

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}.$$

In conclusion, we see that the image domains of the unit disk \mathbb{D} under the functions from \mathcal{F}_β and the operators of such functions are quasidisks. For example, if $f \in \mathcal{F}_1$ then the images $J[f](\mathbb{D})$ and $L[f](\mathbb{D})$ under the Alexander and Libera transforms respectively are quasidisks.

As a last result, we obtain an optimal but not a sharp pre-Schwarzian norm estimates of functions $f \in \mathcal{S}^*(\alpha, \beta)$, $0 < \alpha \leq 1$ and $0 \leq \beta < 1$, of \mathcal{A} , where

$$\mathcal{S}^*(\alpha, \beta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \left(\frac{1 + (1 - 2\beta)z}{1 - z} \right)^\alpha \right\}.$$

Indeed,

$$\|f\| \leq L(\alpha, \beta) + 2\alpha,$$

where

$$L(\alpha, \beta) = \frac{4(1 - \beta)(k - \beta)(k^\alpha - 1)}{(k - 1)(k + 1 - 2\beta)}$$

and k is the unique solution of the following equation in $x \in (1, \infty)$:

$$\begin{aligned} (1 - \alpha)x^{\alpha+2} + \beta(3\alpha - 2)x^{\alpha+1} + [(1 - 2\beta)(1 + \alpha) + 2\beta^2(1 - \alpha)]x^\alpha \\ - \alpha\beta(1 - 2\beta)x^{\alpha-1} - x^2 + 2\beta x = (1 - \beta)^2 + \beta^2. \end{aligned}$$

The sharp estimates for this problem remains an open problem.

1.6. Summary and Conclusion

The current chapter is dedicated for the introduction of some basic concepts and results that we require to present our main results in the sequel. More precisely, we provide some history concerning inequalities and isometries of hyperbolic-type metrics; and the coefficient estimates, radius problems and pre-Schwarzian norm estimates of functions from some subclasses of the class of univalent functions.

In the next three chapters we have studied certain hyperbolic-type metrics such as the Apollonian metric, its inner metric, j metric and its inner metric namely the quasihyperbolic metric. We look at inequalities among them and their geometric interpretation in the sense of constructing or characterizing domains where they hold together. We also obtain isometries of some hyperbolic-type path metrics such as the quasihyperbolic metric, the Ferrand metric and the K–P metric in certain specific domains.

In the remaining two chapters we have considered some geometrically motivated subclasses \mathcal{F} of \mathcal{S} . We obtain the largest disk $|z| < r$ for which $\frac{1}{r}f(rz) \in \mathcal{F}$ whenever $f \in \mathcal{S}$. We also obtain necessary and sufficient coefficient conditions for f to be in \mathcal{F} . In addition, we estimate the pre-Schwarzian norm of functions from \mathcal{F} and that of certain convolution or integral transforms of functions from \mathcal{F} . Some open questions concerning certain classes of univalent functions are studied.

We expect that some of the investigations would lead to new results in different areas of research in function theory.

CHAPTER 2

INEQUALITIES AND GEOMETRY OF THE APOLLONIAN AND RELATED METRICS

The structure of this chapter is as follows. We start by reviewing the definitions, notation and terminology used. The bulk of the chapter consists of five sections which are organized along the different methods used to prove the inequalities in Table 2.1. Specifically, in Section 2.2 we consider the comparison property and uniformity; and in Section 2.3 quasi-isotropy. The main problem in Sections 2.4 and 2.5 is the inequality $\alpha_G \gtrsim \tilde{\alpha}_G$. In Section 2.6 we consider the case when the metrics j_G and $\tilde{\alpha}_G$ are not comparable.

Most of the results of this chapter have been published in: **P. Hästö, S. Ponnusamy and S.K. Sahoo** (2006) Inequalities and geometry of the Apollonian and related metrics. *Rev. Roumaine Math. Pures Appl.* **51**(4), 433–452.

2.1. Introduction

In this chapter we consider the Apollonian metric which was first introduced by Barbilian [4] in 1934–35 and then rediscovered by Beardon [8] in 1998. We also consider the inner metric of the Apollonian metric, the j_G metric and its inner metric, the quasi-hyperbolic metric. We are mainly interested in dealing with inequalities among these metrics (see Table 2.1) and the geometric meaning of these inequalities. The notation used conforms largely to that of [7] and [120], the reader can consult Subsection 2.1.1, if necessary.

Recall that the Apollonian metric is defined for $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ by

$$\alpha_G(x, y) := \sup_{a, b \in \partial G} \log \frac{|a - y| |b - x|}{|a - x| |b - y|}$$

#	Inequality	A	B	#	Inequality	A	B
1.	$\alpha \approx j \approx \tilde{\alpha} \approx k$	+	+	7.	$\alpha \approx j \ll \tilde{\alpha} \ll k$	-	-
2.	$\alpha \ll j \approx \tilde{\alpha} \approx k$	-	-	8.	$\alpha \ll j \ll \tilde{\alpha} \ll k$	-	-
3.	$\alpha \approx j \approx \tilde{\alpha} \ll k$	-	-	9.	$\alpha \approx \tilde{\alpha} \ll j \approx k$	-	+
4.	$\alpha \ll j \approx \tilde{\alpha} \ll k$	-	-	10.	$\alpha \ll \tilde{\alpha} \ll j \approx k$	-	+
5.	$\alpha \approx j \ll \tilde{\alpha} \approx k$	+	+	11.	$\alpha \approx \tilde{\alpha} \ll j \ll k$	-	-
6.	$\alpha \ll j \ll \tilde{\alpha} \approx k$	+	+	12.	$\alpha \ll \tilde{\alpha} \ll j \ll k$	-	-

TABLE 2.1. Inequalities between the metrics α_G , j_G , $\tilde{\alpha}_G$ and k_G . The subscripts are omitted for clarity with the understanding that every metric is defined in the same domain. The A-column refers to whether the inequality can occur in simply connected planar domains, the B-column to whether it can occur in proper subdomains of \mathbb{R}^n .

(with the understanding that $|\infty - x|/|\infty - y| = 1$). This metric was introduced in [8] and has also been considered in [17, 39, 107, 113] and [49]–[66].

For definitions and some of the properties of the Apollonian inner metric, the j_G -metric and the quasihyperbolic metric we refer to Section 1.2.

We will undertake a systematic study of which of the inequalities in (1.8) can hold in the strong form with \ll and which of the relations $j_G \ll \tilde{\alpha}_G$, $j_G \approx \tilde{\alpha}_G$ and $j_G \gg \tilde{\alpha}_G$ can hold. Thus we are led to twelve inequalities, which are given along with the results in Table 2.1, where we have indicated in column A whether the inequality can hold in simply connected planar domains and in column B whether it can hold in an arbitrary proper subdomains of \mathbb{R}^n . From the table we see that most of the cases cannot occur, which means that there are many restrictions on which inequalities can occur together. For instance, we deduce from items 1–4 that $j_G \approx \tilde{\alpha}_G$ implies that $\alpha_G \approx k_G$ and from items 9–12 that the inequality $\tilde{\alpha}_G \ll j_G$ cannot occur in simply connected planar domains.

Since \lesssim is not a linear order, it is also possible that two metrics are not comparable. Therefore we consider separately the case when $j \lesssim \tilde{\alpha}$ in Section 2.6. Since the table does not list this case, one should be careful with the interpretations; for instance, it is

not true that the inequality $\tilde{\alpha}_G \ll k_G$ cannot occur in simply connected planar domains, contrary to what might be thought by considering entries 3, 4, 7, 8, 11 and 12.

2.1.1. Notation

We denote by $\{e_1, e_2, \dots, e_n\}$ the standard basis of \mathbb{R}^n and by n the dimension of the Euclidean space under consideration and assume that $n \geq 2$. For $x \in \mathbb{R}^n$ we denote by x_i its i^{th} coordinate. The following notation is used for Euclidean balls and spheres:

$$B^n(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}, \quad S^{n-1}(x, r) := \{y \in \mathbb{R}^n : |x - y| = r\},$$

$$B^n := B^n(0, 1), \quad S^{n-1} := S^{n-1}(0, 1).$$

We denote by $[x, y]$ the closed segment between x and y .

We use the notation $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one point compactification of \mathbb{R}^n , equipped with the chordal metric. Thus an open ball of $\overline{\mathbb{R}^n}$ as an open Euclidean ball, an open half-space or the complement of a closed Euclidean ball. We denote by ∂G , G^c and \overline{G} the boundary, complement and closure of G , all with respect to $\overline{\mathbb{R}^n}$.

We also need some notation for quantities depending on the underlying Euclidean metric. For $x \in G \subsetneq \mathbb{R}^n$ we write $\delta(x) := d(x, \partial G) := \min\{|x - z| : z \in \partial G\}$. For a path γ in \mathbb{R}^n we denote by $\ell(\gamma)$ its Euclidean length. For $x, y, z \in \mathbb{R}^n$ we denote by \widehat{xyz} the smallest angle between the vectors $x - y$ and $z - y$.

2.2. Basic Inequalities

In this section we define the comparison property and uniformity which are the relations from the introduction that have been most thoroughly studied in the past.

2.2.1. The comparison property

In [53] the term *comparison property* was introduced for the relation $\alpha_G \approx j_G$. Also, an equivalence formulation of this property has been studied by Hästö, see (Theorem 1.3 in [53]).

From the definition of the inner metric (see Section 1.2) it directly follows that if d_1 and d_2 are metrics in the same domain, then $d_1 \approx d_2$ implies that $\tilde{d}_1 \approx \tilde{d}_2$. Therefore Inequalities 3 (Table 2.1), $\alpha_G \approx j_G \approx \tilde{\alpha}_G \ll k_G$, and 7, $\alpha_G \approx j_G \ll \tilde{\alpha}_G \ll k_G$, cannot occur, since in both cases we have assumed the comparison property but not the equivalence of the inner metrics, $\tilde{\alpha}_G$ and k_G .

A well-known fact from [8, Theorem 3.2] is that $\alpha_G \leq 2j_G$ in every domain $G \subsetneq \mathbb{R}^n$. Also, it was shown in [113, Theorem 4.2] that if $G \subsetneq \mathbb{R}^n$ is convex, then $j_G \leq \alpha_G$. So $\alpha_G \approx j_G$ in convex domains.

Lemma 2.1. *Inequality 5, $\alpha_G \approx j_G \ll \tilde{\alpha}_G \approx k_G$, holds in the domain $G := \{x \in \mathbb{R}^n : |x_n| < 1\}$.*

Proof. The domain G is clearly convex, hence it has the comparison property by [113, Theorem 4.2]. From this it follows that $\alpha_G \approx j_G$ and $\tilde{\alpha}_G \approx k_G$. Consider then the points Re_1 and $-Re_1$, where $R > 0$. We have $j_G(Re_1, -Re_1) = \log(1+2R)$ and $k_G(Re_1, -Re_1) = 2R$, hence $j_G \ll k_G$, which concludes the proof. \square

2.2.2. Uniformity

Uniform domains were introduced by O. Martio and J. Sarvas in [80, 2.12], but the following definition is an equivalent form from [40, (1.1)]. In [34], there is a survey of characterizations and implications of uniformity.

Definition 2.2. A domain $G \subsetneq \mathbb{R}^n$ is said to be *uniform* with constant K if for every $x, y \in G$ there exists a path γ , parameterized by arc-length, connecting x and y in G , such that $\ell(\gamma) \leq K|x - y|$; and $K\delta(\gamma(t)) \geq \min\{t, \ell(\gamma) - t\}$.

The relevance of uniformity to our investigation comes from Corollary 1 of [40] which states that a domain G is uniform if and only if $k_G \approx j_G$. This condition is also equivalent to $\tilde{\alpha}_G \lesssim j_G$, see [63, Theorem 1.2]. Thus we have a geometric characterization of domains satisfying this inequality as well.

Example 2.3. The unit ball is uniform and has the comparison property. Hence $\alpha_{B^n} \approx j_{B^n} \approx \tilde{\alpha}_{B^n} \approx k_{B^n}$ and so Inequality 1 can occur.

In fact, Inequality 1 holds in every quasiball, by [49, Corollary 6.9].

Lemma 2.4. *Inequalities 9 (Table 2.1), $\alpha_G \approx \tilde{\alpha}_G \ll j_G \approx k_G$, and 10, $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$ cannot occur in simply connected planar domains.*

Proof. We note that in both these inequalities we have $j_G \approx k_G$ among the assumptions. But a simply connected planar domain is uniform if and only if it is a quasidisk, by [80, Theorem 2.24], and we know that quasidisks have the comparison property, by [53, Corollary 6.3]. Therefore $j_G \approx k_G$ implies that $\alpha_G \approx j_G$ which contradicts $\alpha_G \ll j_G$ in both inequalities. \square

2.3. Quasi-isotropy

We start by introducing some concepts which allow us to calculate the Apollonian inner metric. The concept of quasi-isotropy was introduced in [49] and was studied in depth in [51]. A very similar notion used by Zair Ibragimov is conformality, see [64, 65, 66].

Definition 2.5. We say that a metric space (G, d) with $G \subset \mathbb{R}^n$ is K -quasi-isotropic if

$$\limsup_{r \rightarrow 0} \frac{\sup\{d(x, z) : |x - z| = r\}}{\inf\{d(x, y) : |x - y| = r\}} \leq K$$

for every $x \in G$. A 1-quasi-isotropic metric space is called *isotropic*.

We say that a domain $G \subsetneq \mathbb{R}^n$ is quasi-isotropic if (G, α_G) is K -quasi-isotropic for some constant K ; similarly for isotropic. We define the function qi on the set of proper subdomains of \mathbb{R}^n so that $qi(G)$ is the least constant for which G is quasi-isotropic or $qi(G) = \infty$ if G is not quasi-isotropic for any K . The notion of quasi-isotropy is extended to domains in $\overline{\mathbb{R}^n}$ by Möbius invariance.

Note that the Apollonian metric is not isotropic. It is, nevertheless, possible to define a directed density as follows:

$$\bar{\alpha}_G(x; r) = \lim_{t \rightarrow 0} \frac{1}{t} \alpha_G(x, x + t \frac{r}{|r|}),$$

where $r \in \mathbb{R}^n \setminus \{0\}$. Unless otherwise stated, in this chapter, we will be using the notation r not for a number, but for a vector whenever we talk about the above notation for the directed density. If $\bar{\alpha}_G(x; r)$ is independent of the vector r at every point of G , then the Apollonian metric is isotropic and we may denote $\bar{\alpha}_G(x) := \bar{\alpha}_G(x; e_1)$ and call this function the density of α_G at x . With this concept we can give the following alternative characterization of quasi-isotropy.

Lemma 2.6. [51, Lemma 3.5] *For $G \subsetneq \mathbb{R}^n$ we have*

$$qi(G) = \sup_{x \in G} \frac{\sup_{r \in S^{n-1}} \bar{\alpha}_G(x; r)}{\inf_{r \in S^{n-1}} \bar{\alpha}_G(x; r)},$$

with the understanding that if $\bar{\alpha}_G(x; r) = 0$ for some $x \in G$ and $r \in S^{n-1}$, then $qi(G) = \infty$.

When we do not need the exact value of the quasi-isotropy constant the following lemma is often more convenient to use.

Lemma 2.7. [49, Corollary 5.11] *Let $G \subsetneq \mathbb{R}^n$ be L -quasi-isotropic. Then $\bar{\alpha}_G(x; r)\delta(x) \geq 1/L$ for every $x \in G$ and $r \in S^{n-1}$. If conversely $1/L \leq \bar{\alpha}_G(x; r)\delta(x)$ for every $x \in G$ and $r \in S^{n-1}$, then G is $2L$ -quasi-isotropic.*

In order to present an integral formula for the Apollonian inner metric we need to relate the density of the Apollonian metric with the limiting concept of the Apollonian balls, which we call the Apollonian spheres.

Definition 2.8. Let $G \subsetneq \overline{\mathbb{R}^n}$, $x \in G$ and $\theta \in S^{n-1}$.

- If $B^n(x + s\theta, s) \subset G$ for every $s > 0$ and $\infty \notin G$, then let $r_+ = \infty$.
- If $B^n(x + s\theta, s) \subset G$ for every $s > 0$ and $\infty \in G$, then let r_+ be the largest negative real number such that $G \subset B^n(x + r_+\theta, |r_+|)$.
- Otherwise let $r_+ > 0$ be the largest real number such that $B^n(x + r_+\theta, r_+) \subset G$.

Define r_- in the same way but using the vector $-\theta$ instead of θ . We define the *Apollonian spheres through x in direction θ* by $S_+ := S^{n-1}(x + r_+\theta, r_+)$ and $S_- := S^{n-1}(x - r_-\theta, r_-)$ for finite radii and by the limiting half-space for infinite radii.

Using these spheres we can present a useful result from [49].

Lemma 2.9. [49, Lemma 5.8] *Let $G \subsetneq \overline{\mathbb{R}^n}$ be open, $x \in G \setminus \{\infty\}$ and $\theta \in S^{n-1}$. Let r_{\pm} be the radii of the Apollonian spheres S_{\pm} at x in direction θ . Then*

$$\bar{\alpha}_G(x; \theta) = \frac{1}{2r_+} + \frac{1}{2r_-},$$

where we understand $1/\infty = 0$.

Remark 2.10. The previous lemma was proved in [49] only for the case $G \subsetneq \mathbb{R}^n$. The general case is proved in exactly the same manner.

The following result shows that we can find the Apollonian inner metric by integrating over the directed density, as should be expected. Piecewise continuously differentiable means continuously differentiable except in a finite number of points.

Lemma 2.11. [52, Theorem 1.4] *If $x, y \in G \subsetneq \mathbb{R}^n$, then*

$$\tilde{\alpha}_G(x, y) = \inf_{\gamma} \int \bar{\alpha}_G(\gamma(t); \gamma'(t)) |\gamma'(t)| dt,$$

where the infimum is taken over all paths connecting x and y in G that are piecewise continuously differentiable (with the understanding that $\bar{\alpha}_G(z; 0) = 0$ for all $z \in G$, even though $\bar{\alpha}_G(z; 0)$ is not defined).

The importance of quasi-isotropy to the study of inequalities is a consequence of the following lemma.

Lemma 2.12. [52, Corollary 5.4] *For $G \subsetneq \mathbb{R}^n$ the following conditions are equivalent:*

1. G is quasi-isotropic;
2. $\tilde{\alpha}_G \approx k_G$; and
3. $j_G \lesssim \tilde{\alpha}_G$.

Corollary 2.13. *Inequalities 4 (Table 2.1), $\alpha_G \ll j_G \approx \tilde{\alpha}_G \ll k_G$, and 8, $\alpha_G \ll j_G \ll \tilde{\alpha}_G \ll k_G$, cannot occur.*

Proof. In both cases the assumption $\tilde{\alpha}_G \ll k_G$ implies that $j_G \not\lesssim \tilde{\alpha}_G$, by the previous lemma. This contradicts $j_G \approx \tilde{\alpha}_G$ (in 4) and $j_G \ll \tilde{\alpha}_G$ (in 8). \square

In [49] an exterior ball condition of G was defined as follows: for every $z \in \partial G$ there exists a ball of radius r in the set $G^c \cap B^n(z, Lr)$, where $L > 1$. This condition was shown to be sufficient for the comparison property. The following theorem features a local version of this property.

Theorem 2.14. *Let $G \subsetneq \mathbb{R}^n$ be arbitrary and $L > 1$. For every $x \in G$, let $z \in \partial G$ be such that $|x - z| = \delta(x)$ and suppose there exists a ball B with radius $r_0 = \delta(x)/\sqrt{L^2 - 1}$ such that*

1. $d := d(z, \partial B) \leq r_0(L - 1)$; and
2. for any $y \in B$ the line segment $[x, y]$ connecting x and y intersects ∂G .

Then the inequality $\tilde{\alpha}_G \approx k_G$ holds.

Proof. It follows from Lemma 2.12 that $\tilde{\alpha}_G \approx k_G$ if and only if G is quasi-isotropic. In order to show that G is quasi-isotropic, by Lemma 2.7, it suffices to check that there exists a constant K such that $\bar{\alpha}_G(x; r)\delta(x) \geq 1/K$ for every $x \in G$ and $r \in S^{n-1}$.

Let $x \in G$ and $r \in S^{n-1}$, and fix a ball B as in the statement of the theorem. By (2), we see that the Apollonian spheres with respect to G are smaller in size than with respect to $\mathbb{R}^n \setminus B$ and since $\mathbb{R}^n \setminus B$ is isotropic (as the Apollonian metric equals the hyperbolic metric in a ball) we get

$$\bar{\alpha}_G(x; r) \geq \bar{\alpha}_{\mathbb{R}^n \setminus B}(x; r) = \bar{\alpha}_{\mathbb{R}^n \setminus B}(x) = \frac{1}{\delta(x) + d} - \frac{1}{\delta(x) + d + 2r_0},$$

(the second term is negative, as the corresponding ball contains the point ∞). Now if we use (1) and $r_0 = \delta(x)/\sqrt{L^2 - 1}$, from the hypothesis, then it is easy to estimate that

$$\bar{\alpha}_G(x; r)\delta(x) \geq \frac{2r_0\delta(x)}{(\delta(x) + r_0(L - 1))(\delta(x) + r_0(L + 1))} = \frac{1}{L + \sqrt{L^2 - 1}},$$

and we have a lower bound for $\bar{\alpha}_G(x; r)\delta(x)$. □

The following result provides us with some concrete examples of when the conditions of the previous theorem are satisfied. Although it is intuitively obvious that the examples satisfy the conditions of the theorem, verifying this requires some lengthy calculations and several different cases.

Example 2.15. Let $D \subsetneq \mathbb{R}^2$ be convex and D' be a subset of D which is compact and convex. Let F be a line segment connecting ∂D to $\partial D'$. Then Inequality 6, $\alpha_G \ll j_G \ll \tilde{\alpha}_G \approx k_G$, holds in the domain $G := D \setminus (D' \cup F)$.

Proof. Let $z \in F$ and $\epsilon_0 > 0$ be such that $B^2(z, \epsilon) \subset D \setminus D'$ for all $\epsilon \in (0, \epsilon_0)$. Let $x, y \in S^1(z, \epsilon)$ be diametrically opposite such that $[x, y]$ is perpendicular to F . Then it is easy to see that $\alpha_G(x, y) \rightarrow 0$ as $\epsilon \rightarrow 0$, but on the other hand $j_G(x, y) = \log 3$. Hence $\alpha_G(x, y)/j_G(x, y) \rightarrow 0$ as $\epsilon \rightarrow 0$, which means that $\alpha_G \ll j_G$ holds. Also we note that G is not uniform as it is not possible to connect the same x and y with a path of length comparable to ϵ as $\epsilon \rightarrow 0$, which violates the first condition in Definition 2.2.2 of uniformity. Thus we get $j_G \ll k_G$, because G is uniform if and only if $k_G \approx j_G$ and $j_G \leq k_G$ always holds. We have thus proved that $\alpha_G \ll j_G \ll k_G$. So it remains to prove the last inequality, $\tilde{\alpha}_G \approx k_G$.

Denote $d' := d(D, D')$. Let $B^2(p, r)$ be largest ball contained in D' and $B^2(p, R)$ be the smallest ball with center p containing D' . Since D' is convex and compact, r and R are finite. We define

$$L = \max \left\{ \sqrt{2}, \frac{R}{r}, 1 + \frac{d' + \text{diam } D'}{r} \right\}$$

and check Lemma 2.14 for G with this constant L . For $x \in G$ choose $z \in \partial G$ such that $\delta(x) = |x - z|$. Now, if $z \in \partial D$, take any ball $B \subset D^c$ so that $z \in \partial B$. Then for any $y \in B$ the line segment $[x, y]$ connecting x and y intersects $\partial D \subset \partial G$. Since D is convex we can choose any $L > 1$ in Lemma 2.14 for this x .

Next if $z \in \partial F$, take a line L' perpendicular to F through x and z . Consider the balls with radius $r_0 = \delta(x)/\sqrt{L^2 - 1}$ tangent to both F and L' but on the other side of F than x . Of the two balls satisfying this condition, denote by B the one closer to $F \cap \partial D$. This gives $d(z, \partial B) = r_0(\sqrt{2} - 1)$. Since $L \geq \sqrt{2}$, the hypotheses of Lemma 2.14 are satisfied for this case.

Finally, suppose $z \in \partial D'$. If $\delta(x) \leq r\sqrt{L^2 - 1}$, construct rays L_1 and L_2 starting from z and tangent to $B^2(p, r)$. Choose a ball $B := B^2(w, r_0)$ centered at w and radius $r_0 = \delta(x)/\sqrt{L^2 - 1}$ to which L_1 and L_2 are tangent. Since $r_0 \leq r$, D' is convex and $B \subset D'$, for any $y \in B$ the line segment $[x, y]$ intersects $\partial D' \subset \partial G$. Let a and b be points

where L_1 is tangent to $B^2(p, r)$ and B , respectively. Now it is easy to see that the triangles $\triangle apz$ and $\triangle bwz$ are similar, which gives $d(z, \partial B) \leq r_0(R/r - 1)$, since $|z - p| \leq R$. Because $L \geq R/r$, the hypotheses of Lemma 2.14 are satisfied. If $\delta(x) > r\sqrt{L^2 - 1}$, choose a ball $B \subset D^c$ with radius $r_0 = \delta(x)/\sqrt{L^2 - 1}$ at a distance $d(z, \partial D)$ from z . We see that for any $y \in B$ the line segment $[x, y]$ intersects $\partial D \subset \partial G$. By the triangle inequality, it is clear that $d(z, \partial B) = d(z, \partial D) \leq d' + \text{diam } D'$. Since $L \geq 1 + (d' + \text{diam } D')/r$ and $\delta(x) > r\sqrt{L^2 - 1}$ we get $\delta(x) > \sqrt{(L+1)/(L-1)}(d' + \text{diam } D')$. This gives $d < r_0(L-1)$. Thus for any $z \in \partial G$ with $|x - z| = \delta(x)$, we get all conditions of Lemma 2.14, which gives the conclusion. \square

2.4. Apollonian Quasiconvexity and Comparison Property

In this section we consider Inequalities 2, 11 and 12 (Table 2.1). We prove that none of them can occur in simply connected planar domains and that the first one cannot occur in more general domains, either. Whether the latter two can occur in this case is unclear, although it seems improbable.

We say that a metric space (G, d) is K -*quasiconvex* if for every $x, y \in G$ there exists a path γ connecting x and y in G such that $d(\gamma) \leq Kd(x, y)$, where $d(\gamma)$ is the d -length of γ defined in Section 1.2. We note that the metric d is quasiconvex if and only if $d \approx \tilde{d}$. In [49, Proposition 7.3] it was shown that if α_G is quasiconvex in a simply connected planar domain, then G has the comparison property. Thus $\alpha_G \approx \tilde{\alpha}_G$ implies $\alpha_G \approx j_G$ and so Inequality 11, $\alpha_G \approx \tilde{\alpha}_G \ll j_G \ll k_G$, cannot occur in this case. Let us move on to the other two inequalities.

2.4.1. The twelfth inequality

In this subsection we prove that the inequalities $\alpha_G \ll \tilde{\alpha}_G \ll j_G \ll k_G$ cannot occur in simply connected planar domains. We are not able to establish whether or not it can occur in domains in general. Let us first quote two lemmas from [49].

Lemma 2.16. [49, Lemma 7.1] *Let $G \subset \mathbb{R}^n$ be a domain such that $G \cap B^n = H^n \cap B^n$. Then for every $0 < s < 1$ and every path γ connecting se_n with S^{n-1} we have*

$$\alpha_G(\gamma) \geq \frac{1}{2}(\operatorname{arcsinh} s^{-1} - \operatorname{arcsinh} 1).$$

Lemma 2.17. [49, Lemma 7.2] *Let $G \subsetneq \mathbb{R}^2$ be a simply connected domain and $x, y \in G$ be such that $N\alpha_G(x, y) < j_G(x, y)$ for some $N > 40$. Then there exists a disk $B := B^2(b, r)$ and a unit vector $e \in S^1$ such that*

1. *for all $z \in G^c \cap B$ we have $\langle z - b, e \rangle \leq 4N^{-1/2}r$; and*
2. *the points $b \pm 0.9re$ belong to different path components of $B \cap G$.*

(Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product.)

The proof of the next result is similar to that of Proposition 7.3 in [49].

Proposition 2.18. *If $G \subsetneq \mathbb{R}^2$ is a simply connected domain which does not have the comparison property, then $\tilde{\alpha}_G \not\leq j_G$.*

Proof. Let us assume that G is simply connected but does not have the comparison property. Let $x, y \in G$ be such that $N\alpha_G(x, y) \leq j_G(x, y)$ for some $N > 300$ and define $\epsilon := 2N^{-1/4}$.

Let B be the disk from Lemma 2.17 and assume without loss of generality that $B = B^2$ and $e = e_2$. Let γ be a path connecting ϵe_2 and $-\epsilon e_2$ in G . Every such path passes through S^1 , since it is easy to see that ϵe_2 and $-\epsilon e_2$ are in different components of $B^2 \cap G$.

Let γ_1 be the part of γ in the component of $G \cap B^2$ which contains ϵe_2 . In order to derive a lower bound for the density of the Apollonian metric in γ_1 it suffices to consider the subset $B^2 \cap \partial G$ of the boundary of G . The lower bound gets even smaller if we assume that $B^2 \cap \partial G = \{x \in B^2 : x_2 = -4/\sqrt{N}\}$. We can then apply Lemma 2.16 to γ_1 after using an auxiliary translation ($x \mapsto x + 4e_2/\sqrt{N}$) and scaling ($x \mapsto \sqrt{N}x/\sqrt{N-16}$). Under these operations the point ϵe_2 is mapped to $(\epsilon\sqrt{N} + 4)e_2/\sqrt{N-16}$ and so the lemma applies with $s = (\epsilon\sqrt{N} + 4)/\sqrt{N-16} = (2N^{1/4} + 4)/\sqrt{N-16}$. Thus we find that

$$\tilde{\alpha}_G(\epsilon e_2, -\epsilon e_2) \geq \frac{1}{2} \operatorname{arcsinh} \left(\frac{\sqrt{N-16}}{2N^{1/4} + 4} \right) - \frac{1}{2} \operatorname{arcsinh} 1.$$

On the other hand, we have

$$j_G(\epsilon e_2, -\epsilon e_2) \leq \log(1 + 2\epsilon/(\epsilon - 4/\sqrt{N})) = \log(1 + 2N^{1/4}/(N^{1/4} - 2)).$$

Hence we see that $\tilde{\alpha}_G(\epsilon e_2, -\epsilon e_2)/j_G(\epsilon e_2, -\epsilon e_2) \rightarrow \infty$ as $N \rightarrow \infty$ which means that $\tilde{\alpha}_G \not\lesssim j_G$. \square

The following corollary is immediate.

Corollary 2.19. *Inequalities 11 and 12 (Table 2.1) cannot occur in simply connected planar domains.*

Recall that a quasidisk is the image of a disk under a quasiconformal mapping $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Using the previous result we get yet another characterization of quasidisks (for characterizations in terms of the Apollonian metric see [49], for lots of other characterizations see [33]).

Corollary 2.20. *A simply connected plane domain G is a quasidisk if and only if $\tilde{\alpha}_G \lesssim j_G$.*

Proof. If a simply connected domain G is a quasidisk, then G is uniform by [80, Theorem 2.24] and [80, Corollary 2.33], hence $\tilde{\alpha}_G \lesssim k_G \approx j_G$. Assume conversely that $\tilde{\alpha}_G \lesssim j_G$. It follows from Proposition 2.18 that G has the comparison property and hence also $\tilde{\alpha}_G \approx k_G$ (as in Section 2.2.1). We thus have $k_G \approx \tilde{\alpha}_G \lesssim j_G$, which means that G is uniform and hence a quasidisk by [80, Theorem 2.24]. \square

The following characterization of uniform domains in terms of the Apollonian metric is due to [63].

Lemma 2.21. [63, Theorem 1.2] *A domain D is uniform if and only if there exists a constant c such that $\tilde{\alpha}_D(z_1, z_2) \leq c j_D(z_1, z_2)$ for all $z_1, z_2 \in D$.*

As a consequence, in [63], the authors have established the negation of Inequalities 11 and 12 (Table 2.1) for arbitrary domains, see [63, Corollary 1.3].

2.4.2. The second inequality

In this subsection we prove that Inequality 2, $\alpha_G \ll j_G \approx \tilde{\alpha}_G \approx k_G$, cannot occur in any domain.

Let us quote a lemma from [53] that was used in the proof of Lemma 2.17 which we use to derive a variant of that lemma which is valid in \mathbb{R}^n .

Lemma 2.22. [53, Lemma 3.1] *Let $G \subsetneq \mathbb{R}^n$ be a domain and $x, y \in G$ be points such that $\alpha_G(x, y) \leq j_G(x, y)/N$, for $N \geq 16$. Then there exist balls B , B_1 and B_2 with radii r and $r_1 = r_2 \geq (1 - 3N^{-1/2})r/2$ such that $B_1, B_2 \subset G \cap B$, $d(B_1, B_2) = 2(r - 2r_1)$ and that the segment connecting the centers of B_1 and B_2 intersects ∂G .*

The following corollary is proved from this lemma by considering a sufficiently small ball centered at a boundary point on the segment connecting the centers of B_1 and B_2 .

Corollary 2.23. *If $G \subsetneq \mathbb{R}^n$ does not have the approximation property, then for every $\epsilon > 0$ there exists a point $z \in \partial G$, a real number $r > 0$ and a unit vector $\theta \in S^{n-1}$ such that for every $w \in G^c \cap B^n(z, r)$ we have $\langle w, \theta \rangle \leq \epsilon r$.*

It follows directly from the next theorem that Inequality 2 cannot occur.

Theorem 2.24. *If $G \subsetneq \mathbb{R}^n$ is quasi-isotropic and uniform, then G has the comparison property.*

Proof. Assume that G is L -quasi-isotropic but does not have the comparison property. We will show that this implies that G is not uniform.

Let $0 < \epsilon < 1/(256L^4)$ and choose $u \in \partial G$, $r > 0$ and $e \in S^{n-1}$ such that $\langle v, e \rangle \leq \epsilon r$ for all $v \in G^c \cap B^n(u, r)$ (possible by Corollary 2.23). We assume without loss of generality that $u = 0$, $r = 1$ and $e = e_1$. Consider the points $x := \sqrt{\epsilon}e_1$ and $y := -\sqrt{\epsilon}e_1$ and paths connecting them in G . Let us denote $D := \{z \in B^n(0, \sqrt[4]{\epsilon}) : z_1 = 0\}$ and define A to be the set of paths joining x and y in G which intersect D , and B to be the set of paths joining x and y in G which do not intersect D .

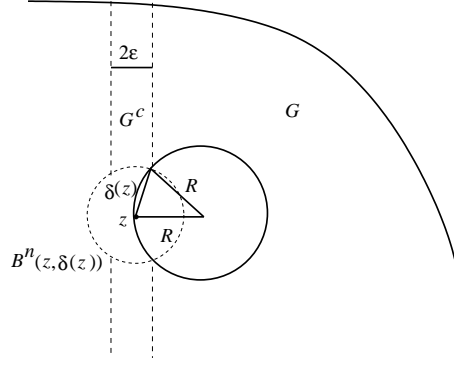


FIGURE 2.1. The Apollonian sphere at z .

Let us consider first a path $\gamma \in A$ parameterized by arc length. Let $z \in \gamma \cap D$, $w \in S^{n-1}(z, \delta(z)) \cap \partial G$ and denote $\theta := (w - z)/|z - w|$. Then $\bar{\alpha}_G(z; \theta) \geq 1/|z - w| = 1/\delta(z)$. Since $B^n(z, \delta(z)) \subset G$ and $G^c \cap B^n$ is contained within ϵ distance from the plane $P := \{\xi \in \mathbb{R}^n : \xi_1 = 0\}$, we find that the Apollonian spheres through z in direction e_1 have radii at least $\min\{\delta(z)^2/(2\epsilon), 1/4\}$, see Figure 2.1. Here the first term comes from spheres limited by the boundary component near the plane P (the case shown in the figure) and the second one comes from spheres limited by S^{n-1} . It follows from this estimate that

$$\frac{\bar{\alpha}_G(z; e_1)}{\bar{\alpha}_G(z; \theta)} \leq \max\{2\epsilon/\delta(z), 4\delta(z)\}.$$

Since G is L -quasi-isotropic, this implies by Lemma 2.6 that $\delta(z) \leq 2L\epsilon$ or $4\delta(z) \geq 1/L$. Since $z \in D$ and $0 \in \partial G$, we have $\delta(z) < \sqrt[4]{\epsilon} < 1/(4L)$ and so we see that the second condition does not hold. This means that $\delta(z) \leq 2L\epsilon$. If t_0 is such that $z := \gamma(t_0) \in D$, then it is clear that $\min\{t_0, \ell(\gamma) - t_0\} \geq d(x, D) = \sqrt{\epsilon}$. Therefore the inequality $K\delta(\gamma(t_0)) \geq \min\{t_0, \ell(\gamma) - t_0\}$, which is the second condition from the definition of uniformity, implies that $K \geq \sqrt{\epsilon}/(2L\epsilon) = 1/(2L\sqrt{\epsilon})$.

On the other hand, for $\gamma \in B$ we have $\ell(\gamma)/|x - y| \geq 1/\sqrt[4]{\epsilon}$. Recall that $\ell(\gamma) \leq K|x - y|$ is the first condition in the definition of uniformity. Thus we see that as $\epsilon \rightarrow 0$ a path γ will violate either the first (if $\gamma \in B$) or the second condition (if $\gamma \in A$) of uniformity, which means that G is not uniform, as was to be shown. \square

Using Theorem 2.24 we can prove the following improvement of Proposition 6.6 from [49] which assumed the comparison property instead of quasi-isotropy in item (2).

Theorem 2.25. *Let $G \subsetneq \mathbb{R}^n$ be a domain. The following conditions are equivalent:*

1. G is A -uniform (i.e. $k_G \lesssim \alpha_G$);
2. G is uniform and quasi-isotropic; and
3. G is Apollonian quasiconvex and quasi-isotropic.

Proof. The three conditions can be written as (1) $\alpha_G \approx k_G$, (2) $j_G \approx k_G$ and $k_G \approx \tilde{\alpha}_G$, and (3) $\tilde{\alpha}_G \approx \alpha_G$ and $k_G \approx \tilde{\alpha}_G$, respectively. If (1) holds, then $\alpha_G \approx j_G \approx \tilde{\alpha}_G \approx k_G$ and it is clear that (2) and (3) hold. Assuming (3) and combining the two inequalities we again get $\alpha_G \approx k_G$, i.e. (1). Finally, if (2) holds, then G has the comparison property by Theorem 2.24, which means that $\alpha_G \approx j_G$ and so $\alpha_G \approx k_G$ and all the metrics are again equivalent. \square

2.5. Apollonian Quasiconvexity, other Constructions

In this section we show that Inequalities 9 (Table 2.1), $\alpha_G \approx \tilde{\alpha}_G \ll j_G \approx k_G$, and 10, $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$, can occur in general domains. Recall that we saw in Lemma 2.4 that these inequalities cannot occur in simply connected planar domains.

2.5.1. The ninth inequality

In this subsection we are especially concerned with the relation $\alpha_G \approx \tilde{\alpha}_G$ (i.e. the question whether or not the Apollonian metric is quasiconvex) to give brief description on Inequality 9. It was shown in [113, Theorem 4.2] $j_G \leq \alpha_G$ for convex G ; hence $\alpha_G \approx j_G$ in convex domains. Recall also the well-known fact that $j_G \approx k_G$ if and only if G is uniform. Thus we conclude that $\alpha_G \approx \tilde{\alpha}_G$ holds for all convex uniform G . In [63, Corollary 1.4] it was shown that $\alpha_G \approx \tilde{\alpha}_G$ implies that G is uniform. On the other hand, there are also domains in which $\alpha_G \ll \tilde{\alpha}_G$; for example, convex domains that are not uniform.

We will now prove the inequality $\alpha_G \approx \tilde{\alpha}_G$ in some set of domains. Unfortunately, we do not have a simple geometric interpretation of this inequality, which means that the proof is somewhat long. However, the structure is simple: first we deal with the “trivial”

cases, where the extra boundary point p has no bearing on the claim. In the other cases we construct a near-geodesic path and estimate its length.

The general idea with the following theorem and its corollary is that the inequality $\alpha_G \approx \tilde{\alpha}_G$ is not disturbed by the addition of some boundary components of co-dimension at least two, but does not hold for the addition of lower co-dimension boundary components.

Theorem 2.26. *Let $D \subsetneq \mathbb{R}^n$ be a bounded domain. Suppose p is a point in D and define $G := D \setminus \{p\}$. If $\alpha_D \approx \tilde{\alpha}_D$, then $\alpha_G \approx \tilde{\alpha}_G$ as well.*

Proof. In this proof we denote by δ the distance to the boundary of D , not of G . We prove $\alpha_G \gtrsim \tilde{\alpha}_G$ since $\alpha_G \leq \tilde{\alpha}_G$ always holds. Let $x, y \in G$ and denote $B := B^n(p, \delta(p)/2)$. Let γ_{xy} be a path connecting x and y such that $\alpha_G(\gamma_{xy}) = \tilde{\alpha}_G(x, y)$. The existence of γ_{xy} is due to [52, Theorem 1.5].

First consider the case $x, y \in D \setminus B$. If $\gamma_{xy} \cap B = \emptyset$, we proceed as follows. Let $z \in \partial G$ be such that $\delta(p) = |p - z|$. Denote $R_D := \text{diam } D / \delta(p)$. For $w \in D \setminus B$ and $r \in S^1$, we have

$$\bar{\alpha}_D(w; r) = \frac{1}{2r_-} + \frac{1}{2r_+} \geq \frac{2}{\text{diam } D} \geq \frac{1}{|w - p|R_D},$$

where the last inequality holds since $|w - p| \geq \delta(p)/2$. We also see that if the Apollonian spheres are affected by the boundary point p , then

$$\bar{\alpha}_G(w; r) \leq \frac{1}{|w - p|} + \frac{1}{2r_+} \leq \frac{1}{|w - p|} + \bar{\alpha}_D(w; r)$$

holds, where r_+ denotes the radius of the Apollonian sphere which touches ∂D . Otherwise, we have $\bar{\alpha}_G(w; r) = \bar{\alpha}_D(w; r)$. So for all $w \in D \setminus B$, the inequalities

$$(2.1) \quad \bar{\alpha}_G(w; r) \leq \bar{\alpha}_D(w; r) + \frac{1}{|w - p|} \leq (1 + R_D)\bar{\alpha}_D(w; r)$$

hold. By Lemma 2.11 we get $\alpha_G(\gamma_{xy}) \leq C\alpha_D(\gamma_{xy})$, for some constant C , which gives

$$(2.2) \quad \tilde{\alpha}_G(x, y) \lesssim \tilde{\alpha}_D(x, y) \approx \alpha_D(x, y) \leq \alpha_G(x, y),$$

where the second inequality holds by assumption and the third holds trivially, as G is a subdomain of D .

If γ_{xy} intersects B , let γ be an intersecting part of γ_{xy} from x_1 to x_2 (if there are more intersecting parts, we proceed similarly). Let γ' be the shortest circular arc on

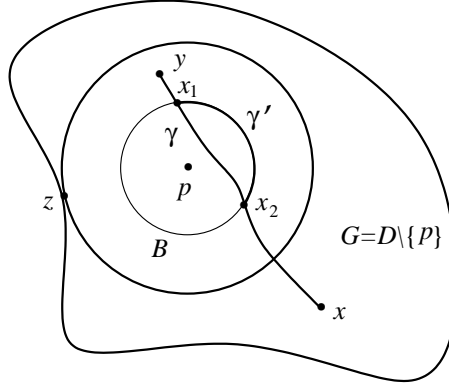


FIGURE 2.2. The geodesic path γ_{xy} of $\tilde{\alpha}_G$ connecting x and y intersects B .

∂B from x_1 to x_2 as shown in the Figure 2.2. Using the density bounds $2/\text{diam } D \leq \bar{\alpha}_D(u; r) \leq 2/\delta(u)$, we see that $\alpha_D(\gamma) \geq 2\ell(\gamma)/\text{diam } D$ and $\alpha_D(\gamma') \leq 4\ell(\gamma')/\delta(p)$ hold. But since $\ell(\gamma) \geq |x_1 - x_2|$ and $\ell(\gamma') \leq \frac{\pi}{2}|x_1 - x_2|$, we have $\ell(\gamma') \lesssim \ell(\gamma)$. This shows that $\alpha_D(\gamma'_{xy}) \lesssim \alpha_D(\gamma_{xy})$ holds. Since $\gamma'_{xy} \subset G \setminus B$, (2.1) implies that $\alpha_G(\gamma'_{xy}) \lesssim \alpha_D(\gamma'_{xy})$. So we get

$$\alpha_G(x, y) \geq \alpha_D(x, y) \approx \tilde{\alpha}_D(x, y) = \alpha_D(\gamma_{xy}) \gtrsim \alpha_D(\gamma'_{xy}) \gtrsim \alpha_G(\gamma'_{xy}) \geq \tilde{\alpha}_G(x, y).$$

Thus we have shown that $\alpha_G(x, y) \gtrsim \tilde{\alpha}_G(x, y)$ holds for all $x, y \in D \setminus B$.

We now consider the case $x, y \in B^n(p, \frac{3}{4}\delta(p))$. Without loss of generality we assume that $|y - p| \leq |x - p|$. Since $\partial G = \partial D \cup \{p\}$, it is clear that

$$\alpha_G(x, y) \geq \max \left\{ \log \frac{|x - p|}{|y - p|}, \alpha_D(x, y) \right\}.$$

Let $\gamma := \gamma_1 \cup \gamma_2$, where γ_1 is the path which is circular about the point p from y to $|y - p|\frac{x-p}{|x-p|} + p$ and γ_2 is the radial part from $|y - p|\frac{x-p}{|x-p|} + p$ to x , as shown in the Figure 2.3. Since the Apollonian spheres are not affected by the boundary point p in the circular part, we have

$$\begin{aligned} \bar{\alpha}_G(\gamma_1(t); \gamma'_1(t)) &\leq \bar{\alpha}_{B^n(p, \delta(p))}(\gamma_1(t); \gamma'_1(t)) = \frac{1}{\delta(p) - |y - p|} + \frac{1}{\delta(p) + |y - p|} \\ &= \frac{2\delta(p)}{\delta(p)^2 - |y - p|^2}, \end{aligned}$$

where the first equality holds since the Apollonian metric is isotropic in balls (since it equals the hyperbolic metric). For $\gamma_2(t)$, by monotony in the domain of definition, we see

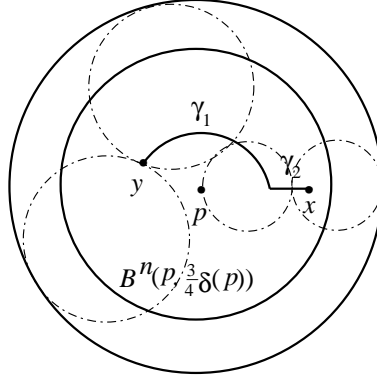


FIGURE 2.3. A short path connecting x and y in $B^n(p, \frac{3}{4}\delta(p))$.

that

$$\begin{aligned} \bar{\alpha}_G(\gamma_2(t); \gamma_2'(t)) &\leq \bar{\alpha}_{B^n(p, \delta(p)) \setminus \{p\}}(\gamma_2(t); \gamma_2'(t)) \\ &= \frac{1}{|p - \gamma_2(t)|} + \frac{1}{\delta(p) - |p - \gamma_2(t)|}. \end{aligned}$$

Hence we have

$$\begin{aligned} \tilde{\alpha}_G(x, y) &\leq \alpha_G(\gamma) \leq \int_{\gamma_1} \frac{2\delta(p)}{\delta(p)^2 - |y - p|^2} |dy| + \int_{|y-p|}^{|x-p|} \frac{1}{t} + \frac{1}{\delta(p) - t} dt \\ &= \frac{2\delta(p)\ell(\gamma_1)}{\delta(p)^2 - |y - p|^2} + \log \left(\frac{|x - p| \delta(p) - |y - p|}{|y - p| \delta(p) - |x - p|} \right) \\ &\leq \frac{32}{7} \frac{\ell(\gamma_1)}{\delta(p)} + \log \left(\frac{|x - p| \delta(p) - |y - p|}{|y - p| \delta(p) - |x - p|} \right). \end{aligned}$$

Since $u \mapsto u^3(\delta(p) - u)$ is increasing for $0 < u < 3\delta(p)/4$ and we have $|y - p| \leq |x - p| \leq 3\delta(p)/4$, the inequality $|x - p|^3(\delta(p) - |x - p|) \geq |y - p|^3(\delta(p) - |y - p|)$ holds. This inequality is equivalent to

$$\log \left(\frac{|x - p| \delta(p) - |y - p|}{|y - p| \delta(p) - |x - p|} \right) \leq 4 \log \frac{|x - p|}{|y - p|}.$$

Using $\alpha_D \approx \tilde{\alpha}_D$ we easily get $\alpha_D(x, y) \gtrsim \ell(\gamma_1)/\delta(p)$. We have thus shown that

$$\tilde{\alpha}_G(x, y) \leq K\alpha_D(x, y) + 4 \log\{|x - p|/|y - p|\} \leq (K + 4)\alpha_G(x, y),$$

for some constant K .

It remains to consider the case $x \notin B^n(p, 3\delta(p)/4)$ and $y \in B$. Let $w \in S^{n-1}(p, \frac{3}{4}\delta(p))$ be such that $|y - w| = d(y, S^{n-1}(p, \frac{3}{4}\delta(p)))$. Let $\gamma := \gamma_1 \cup \gamma_2$, where $\gamma_1 = [y, w]$ and γ_2 is

a path connecting w and x such that $\alpha_G(\gamma_2) = \tilde{\alpha}_G(x, w)$. As we discussed in the previous case, we have

$$\alpha_G(\gamma_1) \leq 4 \log \frac{3\delta(p)}{4|y-p|} \leq 4 \log \frac{|x-p|}{|y-p|} \leq 4\alpha_G(x, y).$$

Since $x, w \notin B$, it follows by the previous cases that

$$\alpha_G(\gamma_2) = \tilde{\alpha}_G(w, x) \lesssim \alpha_D(w, x) \leq 2j_D(w, x).$$

If $\delta(w) \leq \delta(x)$, using the first inequality of (2.2) and the triangle inequality we see that the inequalities

$$\alpha_G(\gamma_2) = \tilde{\alpha}_G(w, x) \lesssim \tilde{\alpha}_D(w, x) \approx \alpha_D(w, x) \leq 2j_D(w, x) \leq 2 \log \left(4 + \frac{4|x-p|}{\delta(p)} \right)$$

hold. But for $s \geq 3/2$, we have $\log(4 + 2s) \leq 5 \log s$. Since $|y-p| \leq \delta(p)/2$, we obtain

$$\alpha_G(\gamma_2) \lesssim \log \left(4 + \frac{4|x-p|}{\delta(p)} \right) \leq 5 \log \frac{|x-p|}{|y-p|} \leq 5\alpha_G(x, y).$$

We then move on to the case $\delta(w) \geq \delta(x)$. If $|x-y| \geq 3\delta(x)$, we see (by the triangle inequality) that

$$\alpha_G(x, y) \geq \sup_{b \in \partial D} \log \frac{|x-y| - |b-x|}{|b-x|} = \log \left(\frac{|x-y|}{\delta(x)} - 1 \right)$$

holds. Using this and the fact that $\frac{|x-p|}{|y-p|} \geq \log(3/2)$ we get

$$\alpha_G(x, y) \geq \begin{cases} \log \left(\frac{|x-y|}{\delta(x)} - 1 \right) & \text{for } \frac{|x-y|}{\delta(x)} \geq 3, \\ \log \frac{3}{2} & \text{otherwise.} \end{cases}$$

Since $|x-y| \geq \delta(p)/4$, we get the following upper bound for the length of the curve (the first inequality follows as before):

$$\alpha_G(\gamma_2) \lesssim j_D(x, w) \lesssim \log \left(1 + \frac{|x-y| + \delta(p)}{\delta(x)} \right) \leq \log \left(1 + \frac{5|x-y|}{\delta(x)} \right).$$

The function $f(s) = (s-1)^4 - (1+5s)$ is increasing for $s \geq 3$, so $f(s) \geq f(3) = 0$. Thus for $|x-y|/\delta(x) \geq 3$, we get

$$\alpha_G(\gamma_2) \lesssim \log \left(1 + \frac{5|x-y|}{\delta(x)} \right) \leq 4 \log \left(\frac{|x-y|}{\delta(x)} - 1 \right) \leq 4\alpha_G(x, y).$$

On the other hand, if $|x-y|/\delta(x) < 3$, then $\alpha_G(\gamma_2)$ is bounded above by $4 \log 2$ and $\alpha_G(x, y)$ is bounded below by $\log(3/2)$, so the inequality $\alpha_G(\gamma_2) \lesssim \alpha_G(x, y)$ is clear. We have now verified the inequality in all the possible cases, so the proof is complete. \square

Corollary 2.27. *Let $D \subsetneq \mathbb{R}^n$ be a bounded domain. Suppose $(p_i)_{i=1}^k$ is a finite non-empty sequence of points in D and define $G := D \setminus \{p_1, p_2, \dots, p_k\}$. Assume that $\alpha_D \approx \tilde{\alpha}_D$ and $j_D \approx k_D$. Then Inequality 9, $\alpha_G \approx \tilde{\alpha}_G \ll j_G \approx k_G$, holds.*

Proof. Since $j_D \approx k_D$, D is uniform and thus so G is, as can be seen from the definition, which implies that $j_G \approx k_G$ holds. Let $\epsilon_0 > 0$ be such that the sphere $S^{n-1}(p_1, \epsilon) \subset D$ for all $\epsilon \in (0, \epsilon_0)$. Let $x, y \in S^{n-1}(p_1, \epsilon)$ be diametrically opposite. Then we see that $\alpha_G(x, y) \rightarrow 0$ as $\epsilon \rightarrow 0$, but on the other hand $j_G(x, y) = \log 3$. Hence $\alpha_G(x, y)/j_G(x, y) \rightarrow 0$ as $\epsilon \rightarrow 0$, which implies that $\alpha_G \ll j_G$. We have thus proved that $\alpha_G \ll j_G \approx k_G$. So, it remains to prove $\alpha_G \gtrsim \tilde{\alpha}_G$, since $\alpha_G \leq \tilde{\alpha}_G$ always holds. For $1 \leq i \leq k$, define $G_i = G_{i-1} \setminus \{p_i\}$, where $G_0 = D$. Since D is bounded and $\alpha_D \approx \tilde{\alpha}_D$, we conclude by Theorem 2.26 that $\alpha_{G_1} \approx \tilde{\alpha}_{G_1}$. Inductively, we get $\alpha_{G_i} \approx \tilde{\alpha}_{G_i}$ for all i , $1 \leq i \leq k$. Since $G_k = D \setminus \{p_1, \dots, p_k\} = G$, we have shown that $\alpha_G \approx \tilde{\alpha}_G$. \square

One ingredient in the proofs of some of the inequalities in Theorem 2.26 was the following reformulated result, which shows that removing a point from the domain (i.e. adding a boundary point) does not affect the inequality $\alpha_G \approx \tilde{\alpha}_G$:

Theorem 2.28. *Let $D \subsetneq \mathbb{R}^n$ be a domain with an exterior point. Let $p \in D$ and $G := D \setminus \{p\}$. If $\alpha_D \approx \tilde{\alpha}_D$, then $\alpha_G \approx \tilde{\alpha}_G$ as well.*

Note that by Möbius invariance, one may assume that the exterior point is in fact ∞ , in which case the domain is bounded, as was the assumption in the original source. This assumption was of a technical nature, and in the following result we show that indeed it can be replaced by a much weaker assumption that the complement of D is not contained in a hyperplane. This assumption is sharp, for if $\partial G \subset \mathbb{R}^{n-1}$, then $\alpha_G(x, \tilde{x}) = 0$, where \tilde{x} is the reflection of x in \mathbb{R}^{n-1} , but $\tilde{\alpha}_G(x, \tilde{x}) > 0$, so that $\alpha_G \not\approx \tilde{\alpha}_G$ and the theorem becomes vacuous.

Theorem 2.29. *Let $D \subsetneq \mathbb{R}^n$ be a domain whose boundary is not contained in a hyperplane. Let $p \in D$ and $G := D \setminus \{p\}$. If $\alpha_D \approx \tilde{\alpha}_D$, then $\alpha_G \approx \tilde{\alpha}_G$ as well.*

Proof. In this proof we denote by δ the distance to the boundary of D , but not of G . It is enough to prove the inequality $\alpha_G \gtrsim \tilde{\alpha}_G$, because other way inequality always holds.

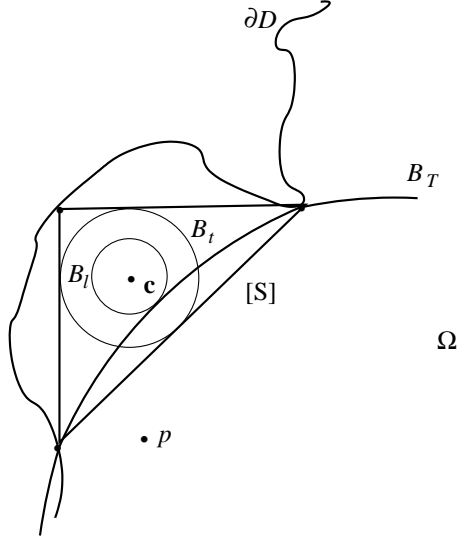


FIGURE 2.4. The largest ball B_T tangent to B_l and contained in $\Omega = \mathbb{R}^n \setminus S$.

Let $x, y \in G$ and denote $B := B^n(p, \delta(p)/2)$. Let γ_{xy} be a path connecting x and y such that $\alpha_G(\gamma_{xy}) = \tilde{\alpha}_G(x, y)$, see the definition of the inner metric.

Let us first consider the case when $x, y \in D \setminus B$ and $\gamma_{xy} \cap B = \emptyset$.

Let $z \in \partial D$ be such that $\delta(p) = |p - z|$. Let S be the collection of $n + 1$ boundary points of D where they form the vertices of an n -simplex. Denote by $B_t := B^n(c, t)$ the largest ball with radius t and centered at c such that B_t is inside the n -simplex $[S]$. Define $l = t/2$. Denote by $B_l := B^n(c, l)$ the ball with radius l and centered at c . Define $\Omega = \mathbb{R}^n \setminus S$. Let $B_T \subset \Omega$ be the largest ball with radius T and tangent to B_l , see Figure 2.4.

Choose $L = 5 \max\{|p - c|, T\}$. Consider the ball $B^n(c, L)$ centered at c with radius L and denote it by B_L . Then we see that $S \cup \{\infty\} \subset \partial D$. Since

$$\partial\Omega = \partial(\mathbb{R}^n \setminus S) = S \cup \{\infty\} \subset \partial D,$$

we see that

$$(2.3) \quad \bar{\alpha}_D(w; r) \geq \bar{\alpha}_\Omega(w; r)$$

for $r \in S^{n-1}$.

For the moment we assume $w \in \mathbb{R}^n \setminus B_L$. We now estimate the density of the Apollonian spheres in Ω passing through w and in the direction r . We denote F the ray from w along the direction of r . Consider a sphere S_1 with radius R_1 and centered at $x \in F$ such that S_1 is tangent to B_l . Denote $\theta := \widehat{xwc}$. Construction of B_T gives that, for $|\theta| < \pi/2$ we see that the Apollonian spheres passing through w and in the direction r are smaller in size than the sphere S_1 . This gives

$$(2.4) \quad \bar{\alpha}_\Omega(w; r) \geq \frac{1}{2R_1} = \frac{l + (d+l) \cos \theta}{(d+l)^2 - l^2},$$

where $d := d(w, B_l)$ and R_1 is obtained using the cosine formula for the triangle Δxwc .

Now the ball with radius R_2 and center at q passing through w and p gives

$$\frac{|w-p|}{2} = R_2 \cos(\theta - \psi),$$

where $\psi = \widehat{cwp}$ and $q \in F$. If the Apollonian spheres are affected by the boundary point p then

$$\begin{aligned} \bar{\alpha}_G(w; r) &= \frac{1}{R_2} + \frac{1}{r_+} \\ &\leq \frac{1}{R_2} + \bar{\alpha}_D(w; r) \\ &= \frac{2 \cos(\theta - \psi)}{|w-p|} + \bar{\alpha}_D(w; r) \end{aligned}$$

hold, where r_+ denotes the radius of the smaller Apollonian sphere which touches ∂D .

Denote $\phi := \widehat{wpc}$. Since $p \in B_L$, using the sine formula in the triangle Δwpc we get

$$\sin \psi = \frac{|p-c|}{|w-p|} \sin \phi \leq \frac{|p-c|}{|w-p|}.$$

Then we see that

$$\cos(\theta - \psi) \leq \cos \theta + \sin \psi \leq \cos \theta + \frac{|p-c|}{|w-p|}.$$

Thus we get

$$\begin{aligned} \bar{\alpha}_G(w; r) &\leq \frac{2(\cos \theta + |p-c|/|w-p|)}{|w-p|} + \bar{\alpha}_D(w; r) \\ &= \frac{2 \cos \theta}{|w-p|} + \frac{2|p-c|}{|w-p|^2} + \bar{\alpha}_D(w; r). \end{aligned}$$

Since $w \notin B_L$, triangle inequality gives $|w - p| \approx d$. We then obtain

$$\begin{aligned}
 \bar{\alpha}_G(w; r) &\lesssim \frac{\cos \theta}{d} + \frac{l}{d^2} + \bar{\alpha}_D(w; r) \\
 (2.5) \qquad &\approx \frac{l + (d + l) \cos \theta}{(d + l)^2 - l^2} + \bar{\alpha}_D(w; r).
 \end{aligned}$$

Now we see that if $\bar{\alpha}_\Omega(w; r) = 0$ then $\partial\Omega$ is contained in a hyperplane, which contradicts our assumption. Thus if $w \in \overline{B_L}$ then $\bar{\alpha}_\Omega(w; r) > 0$ and since the density function is continuous it has a greatest lower bound, i.e. there exists a constant $k > 0$ such that for $r \in S^1$ we have

$$(2.6) \qquad \bar{\alpha}_\Omega(w; r) \geq k.$$

Therefore, (2.4) and (2.6) together give

$$(2.7) \qquad \bar{\alpha}_\Omega(w; r) \geq \min \left\{ \frac{l + (d + l) \cos \theta}{(d + l)^2 - l^2}, k \right\}.$$

In B_L , we also see that if the Apollonian spheres are affected by the boundary point p then

$$\begin{aligned}
 \bar{\alpha}_G(w; r) &\leq \frac{1}{|w - p|} + \frac{1}{2r_+} \\
 &\leq \frac{1}{|w - p|} + \bar{\alpha}_D(w; r) \\
 &\leq \frac{2}{\delta(p)} + \bar{\alpha}_D(w; r) \\
 (2.8) \qquad &\approx k + \bar{\alpha}_D(w; r),
 \end{aligned}$$

where r_+ denotes the radius of the smaller Apollonian sphere which touches ∂D . Thus (2.3), (2.5), (2.7) and (2.8) give

$$\begin{aligned}
 \bar{\alpha}_G(w; r) &\lesssim \min \left\{ \frac{l + (d + l) \cos \theta}{(d + l)^2 - l^2}, k \right\} + \bar{\alpha}_D(w; r) \\
 (2.9) \qquad &\lesssim \bar{\alpha}_D(w; r).
 \end{aligned}$$

Hence we get the relation $\alpha_G(\gamma_{xy}) \leq K \alpha_D(\gamma_{xy})$ for some constant K . This gives

$$(2.10) \qquad \tilde{\alpha}_G(x, y) \lesssim \tilde{\alpha}_D(x, y) \approx \alpha_D(x, y) \leq \alpha_G(x, y),$$

where the second inequality holds by assumption and the third holds trivially, as G is a subdomain of D .

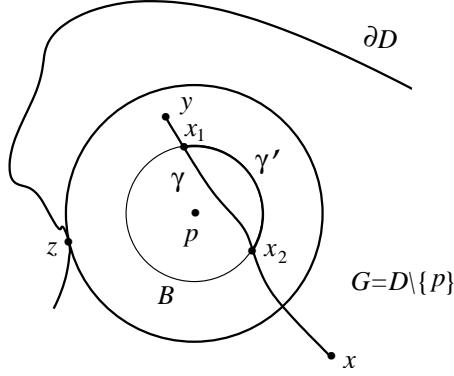


FIGURE 2.5. The geodesic path γ_{xy} of $\tilde{\alpha}_G$ connecting x and y intersects B .

Let us now consider the case when $x, y \in D \setminus B$ and γ_{xy} intersects B .

Let γ be an intersecting part of γ_{xy} from x_1 to x_2 (if there are more intersecting parts, we proceed similarly). Let γ' be the shortest circular arc on ∂B from x_1 to x_2 , as shown in the Figure 2.5.

Using the density bounds (2.3) and (2.6) we get $k \leq \bar{\alpha}_D(u; r) \leq 2/\delta(u)$. Then we see that

$$\alpha_D(\gamma) \geq \ell(\gamma)/k \quad \text{and} \quad \alpha_D(\gamma') \leq 4\ell(\gamma')/\delta(p)$$

hold. But since $\ell(\gamma) \geq |x_1 - x_2|$ and $\ell(\gamma') \leq \frac{\pi}{2}|x_1 - x_2|$, we have $\ell(\gamma') \lesssim \ell(\gamma)$. This shows that $\alpha_D(\gamma'_{xy}) \lesssim \alpha_D(\gamma_{xy})$ holds. Since $\gamma'_{xy} \subset G \setminus B$, (2.9) implies that $\alpha_G(\gamma'_{xy}) \lesssim \alpha_D(\gamma'_{xy})$. So we get

$$\alpha_G(x, y) \geq \alpha_D(x, y) \approx \tilde{\alpha}_D(x, y) = \alpha_D(\gamma_{xy}) \gtrsim \alpha_D(\gamma'_{xy}) \gtrsim \alpha_G(\gamma'_{xy}) \geq \tilde{\alpha}_G(x, y).$$

Thus we have shown that $\alpha_G(x, y) \gtrsim \tilde{\alpha}_G(x, y)$ holds for all $x, y \in D \setminus B$.

Proof of rest of the cases are same as in that of Theorem 2.26. \square

Of course, we can iterate Theorem 2.29, to remove any finite set of points from our domain. Exactly like in Corollary 2.27, we get

Corollary 2.30. *Let $D \subsetneq \mathbb{R}^n$ be a domain whose boundary does not lie in a hyperplane. Suppose $(p_i)_{i=1}^k$ is a finite non-empty sequence of points in D and define $G := D \setminus \{p_1, p_2, \dots, p_k\}$. Assume that $\alpha_D \approx \tilde{\alpha}_D$ and $j_D \approx k_D$. Then Inequality 9, $\alpha_G \approx \tilde{\alpha}_G \ll j_G \approx k_G$, holds.*

2.5.2. The tenth inequality

In this subsection we construct a domain in \mathbb{R}^3 which is topologically equivalent (in $\overline{\mathbb{R}^3}$) to a ball in which the inequalities $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$ hold.

Proposition 2.31. *Define $R_1 := \{e_1 + te_3 : t \in [0, \infty)\}$ and $R_2 := \{-e_1 + te_3 : t \in [0, \infty)\}$. In the domain*

$$G := \mathbb{R}^3 \setminus \left(R_1 \cup R_2 \cup \overline{B^3}(e_1 - e_3, 1) \right)$$

Inequality 10, $\alpha_G \ll \tilde{\alpha}_G \ll j_G \approx k_G$, holds.

Proof. It is easy to see that G is uniform, if we handle the cases when $|x - y|$ is small and when it is large separately. In the former case, both x and y are near a single boundary component of G and hence we need only consider the boundary components separately. If $|x - y|$ is large, then we may choose the path to curve away from all boundary components. Since G contains a boundary component which is a ray, it is clear that G is not quasi-isotropic, hence $\tilde{\alpha}_G \ll k_G$ by Lemma 2.12 and it remains only to prove that $\alpha_G \ll \tilde{\alpha}_G$.

Set $x := te_3 + 2e_2$ and $y := te_3 - 2e_2$ for $t \in [3/2, \infty)$. It is clear that $\alpha_G(x, y) \rightarrow 0$ as $t \rightarrow \infty$, since the rays R_1 and R_2 do not affect this distance. We next derive a lower bound for $\tilde{\alpha}_G(x, y)$ which is independent of t . In so doing we may forget about the boundary points in the sphere $S^2(e_1 - e_3, 1)$ since this only makes the bound smaller. Denote $B^x := B^3(x, 1)$ and $B^y := B^3(y, 1)$. Let $u \in B^x$, $\theta \in S^1$ and denote $d := d(u, \partial B^y)$. We see that any ball which intersects B^x and B^y will also intersect R_1 or R_2 . Thus the Apollonian spheres through u in direction θ with respect to G are smaller in size than with respect to $\mathbb{R}^3 \setminus B^y$. Since $\mathbb{R}^3 \setminus B^y$ is isotropic, we get

$$\bar{\alpha}_G(u; \theta) \geq \bar{\alpha}_{\mathbb{R}^3 \setminus B^y}(u; \theta) = \bar{\alpha}_{\mathbb{R}^3 \setminus B^y}(u) = \frac{1}{d} - \frac{1}{d+2} \geq \frac{1}{12},$$

where we used $d \leq 4$ for the last inequality (the minus sign occurs because the corresponding Apollonian sphere contains ∞). Let γ be a path connecting x and y . Then it certainly connects x to ∂B^x ; denote this part by γ' . By Theorem 2.11 we get $\tilde{\alpha}_G(x, y) \geq \frac{1}{12} \inf_{\gamma'} \ell(\gamma') = \frac{1}{12}$. Since $\alpha_G(x, y) \rightarrow 0$ as $t \rightarrow \infty$, we see that $\alpha_G \ll \tilde{\alpha}_G$. \square

2.6. Noncomparability

In this short section we sort out the possible inequalities when j_G and $\tilde{\alpha}_G$ are not comparable. It turns out that there is just one possibility in this case. For if $j_G \leq \tilde{\alpha}_G$, then it follows without any geometrical considerations that none of the inequalities $\alpha_G \approx j_G$, $\alpha_G \approx \tilde{\alpha}_G$, $j_G \approx k_G$ or $\tilde{\alpha}_G \approx k_G$ can hold, since if for instance $\alpha_G \approx j_G$, then $j_G \approx \alpha_G \lesssim \tilde{\alpha}_G$, contrary to assumption. Hence only the possibility $\alpha_G \ll j_G \ll k_G$ and $\alpha_G \ll \tilde{\alpha}_G \ll k_G$ remains, which is the case of least possible comparability among the metrics. Unfortunately, this occurs in quite many domains.

Proposition 2.32. *Let G be a simply connected planar domain which is not quasi-isotropic. Then $\alpha_G \ll j_G \ll k_G$, $\alpha_G \ll \tilde{\alpha}_G \ll k_G$ and $j_G \leq \tilde{\alpha}_G$.*

Proof. Since G is not quasi-isotropic, we have $j_G \not\asymp \tilde{\alpha}_G$ by Corollary 2.12. Since G is simply connected and does not have the comparison property, the inequality $j_G \not\asymp \tilde{\alpha}_G$ follows from Proposition 2.18. Hence $j_G \leq \tilde{\alpha}_G$ which implies the inequalities $\alpha_G \ll j_G \ll k_G$ and $\alpha_G \ll \tilde{\alpha}_G \ll k_G$, as was shown above. \square

Example 2.33. The domain $H^2 \setminus [0, e_2]$ satisfies the assumptions of the previous lemma.

CHAPTER 3

UNIFORM, JOHN AND QUASI-ISOTROPIC DOMAINS

This chapter concerns characterization of uniform domains in terms of the lower bound of the λ -Apollonian metric. In addition, we consider relationship between quasi-isotropic domains and quasidisks. In Section 3.1, we recall a number of results which motivate us to investigate the main contexts of this chapter. This section also includes basic definitions, notational descriptions, and some elementary results for proving our main results. In Section 3.2, we state and prove a few technical lemmas and as a consequence, we establish the proof of Theorem 3.8 and solutions to a number of related questions in terms of examples. Finally, Section 3.3 is devoted to discussions on relationship between simply connected quasi-isotropic domains and John disks.

Results of this chapter are from the published article: **M. Huang, X. Wang, S. Ponnusamy and S.K. Sahoo** (2008) Uniform domains, John domains and quasi-isotropic domains. *J. Math. Anal. Appl.* **343**, 110–126.

3.1. Introduction and Preliminaries

Throughout the chapter, we always assume that D is a proper subdomain of the complex plane \mathbb{C} possessing at least two finite boundary points, and that constants such as b and c are positive.

As in [83], a simply connected domain D is called a b -John disk if for any two points $z_1, z_2 \in D$, there is a rectifiable arc $\gamma \subset D$ joining them with

$$\min_{j=1,2} \ell(\gamma[z_j, z]) \leq b \operatorname{dist}(z, \partial D)$$

for all $z \in \gamma$, where b is a constant. Sometimes we simply call D a *John disk* if it is a b -John disks for some positive constant b .

The class of so-called John domains and uniform domains enjoy an important role in many areas of modern mathematical analysis, see [80, 83, 118]. Martio and Sarvas [80] were the first who introduced uniform domains and since then its importance along with John domains throughout the function theory is well documented, see [34, 118]. It is well-known that a simply connected planar domain D is a quasidisk if and only if D is a uniform domain (see [49, Lemma 6.4]); a Jordan domain D is a quasidisk if and only if both D and $D^* := \mathbb{C} \setminus \overline{D}$ are John disks, and every quasidisk is a John disk (see [69]). Hence John disks can be thought of as “one-sided quasidisks”.

For any $z_1, z_2 \in D$, the λ -length [24, 25] between them is defined by

$$\lambda_D(z_1, z_2) = \inf\{\ell(\gamma) : \gamma \subset D \text{ is a rectifiable arc joining } z_1 \text{ and } z_2\}.$$

A point w in the boundary ∂D of D is said to be *rectifiably accessible* if there is a half open rectifiable arc γ in D ending at w . Let $\partial_r D$ denote the subset of ∂D which consists of all the rectifiably accessible points, that is

$$\partial_r D = \{w \in \partial D : w \text{ is rectifiably accessible}\}.$$

The λ -Apollonian metric α'_D (see [25, 121]) is defined by

$$\alpha'_D(z_1, z_2) = \sup_{w_1, w_2 \in \partial_r D} \log |z_1, z_2, w_1, w_2|_{\lambda_D}.$$

Here

$$|z_1, z_2, w_1, w_2|_{\lambda_D} = \frac{\lambda_D(z_1, w_1)\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)\lambda_D(z_2, w_1)}$$

for all $z_1, z_2 \in D$.

Lemma 3.1. *For all $z_1, z_2 \in D$ we have*

$$\alpha_D(z_1, z_2) \leq \alpha'_D(z_1, z_2).$$

Proof. From the properties of the Apollonian balls approach (see Subsection 1.1.1) we have

$$(3.1) \quad \sup_{w_1 \in \partial D} \frac{|z_1 - w_1|}{|z_2 - w_1|} \geq 1,$$

for $z_1, z_2 \in D$. Suppose $w \in \partial D$ is a point such that

$$(3.2) \quad \sup_{w_1 \in \partial D} \frac{|z_1 - w_1|}{|z_2 - w_1|} = \frac{|z_1 - w|}{|z_2 - w|}.$$

Then we must have

$$\lambda_D(z_2, w) = |z_2 - w|.$$

For a proof, we assume a contradiction. Then there exists a point w' (different from w) in ∂D such that

$$(3.3) \quad |z_2 - w| = |z_2 - w'| + |w' - w|$$

with $\lambda_D(z_2, w') = |z_2 - w'|$. Now, by the triangle inequality, we see that

$$|z_1 - w'| \geq |z_1 - w| - |w - w'| = |z_1 - w| - |z_2 - w| + |z_2 - w'|.$$

This gives that

$$\begin{aligned} \frac{|z_1 - w'|}{|z_2 - w'|} &\geq 1 + \frac{|z_1 - w| - |z_2 - w|}{|z_2 - w'|} \\ &> 1 + \frac{|z_1 - w| - |z_2 - w|}{|z_2 - w|} \\ &= \frac{|z_1 - w|}{|z_2 - w|}, \end{aligned}$$

where the strict inequality holds by (3.1) and (3.3), which is a contradiction due to (3.2).

Thus, we conclude that

$$(3.4) \quad \sup_{w_1 \in \partial D} \frac{|z_1 - w_1|}{|z_2 - w_1|} \leq \sup_{w_1 \in \partial_r D} \frac{\lambda_D(z_1, w_1)}{\lambda_D(z_2, w_1)},$$

because $|z_1 - w_1| \leq \lambda_D(z_1, w_1)$ always holds. Similarly, we obtain

$$(3.5) \quad \sup_{w_2 \in \partial D} \frac{|z_2 - w_2|}{|z_1 - w_2|} \leq \sup_{w_2 \in \partial_r D} \frac{\lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2)}.$$

Relations (3.4) and (3.5) together give the required conclusion. \square

Recall that the symbol $\delta(z)$ stands for $\text{dist}(z, \partial D)$, the Euclidean distance from z to the boundary ∂D of D . Also as in [40], for $z_1, z_2 \in D$ the metric $j_D(z_1, z_2)$ is defined by

$$j_D(z_1, z_2) = \log \left(1 + \frac{|z_1 - z_2|}{\delta(z_1)} \right) \left(1 + \frac{|z_1 - z_2|}{\delta(z_2)} \right).$$

The metric j'_D is obtained by replacing the Euclidean distance in the definition of j_D metric by the λ -length (c.f. [25]).

The following Bernoulli inequalities are crucial to prove certain inequalities in our context:

Lemma 3.2. *For $x \geq 0$, we have*

$$\log(1 + cx) \leq c \log(1 + x) \text{ if } c \geq 1$$

and

$$\log(1 + cx) \geq c \log(1 + x) \text{ if } 0 \leq c \leq 1.$$

These two inequalities follow, for example, from the fact that $c \mapsto \log(1 + cx)/c$ is a decreasing function of c in $(0, \infty)$, for each fixed $x > 0$.

We denote by xy the line through x and y , and by $(x, y]$ the semi closed (or semi open) segment between x and y .

We begin our discussion with the following simple characteristic property of quasidisks due to Gehring.

Theorem A [35, Theorem 11]. *A simply connected domain D is a quasidisk if and only if there is a constant c such that*

$$h_D(z_1, z_2) \leq c j_D(z_1, z_2)$$

for all $z_1, z_2 \in D$.

Later in 2000, Gehring and Hag obtained the following interesting characterization.

Theorem B [39, Theorem 3.1]. *A simply connected domain D is a quasidisk if and only if there is a constant c such that*

$$h_D(z_1, z_2) \leq c \alpha_D(z_1, z_2)$$

for all $z_1, z_2 \in D$.

Next, we recall the following simple and useful characteristic properties of John disks due to Näkki and Väisälä [83].

Theorem C [83]. *Let D be a simply connected proper subdomain in \mathbb{C} . Then the following conditions are equivalent:*

- (1) D is a b -John disk;
- (2) For each $z \in \mathbb{R}^2$ and $r > 0$, any two points in $D \setminus \overline{\mathbb{D}}(z, r)$ can be joined by an arc in $D \setminus \overline{\mathbb{D}}(z, \frac{r}{c})$, where the constants b and c depend only on each other and $\mathbb{D}(z, r)$ denotes the open disk of radius r centered at the point z ;
- (3) For every straight crosscut γ of D dividing D into subdomains D_1 and D_2 , we have $\min_{j=1,2} \text{diam}(D_j) \leq c \text{diam}(\gamma)$, where the constants b and c depend only on each other and $\text{diam}(\gamma)$ means the diameter of γ .

By using h_D and j'_D , Kim and Langmeyer obtained the following necessary and sufficient conditions for b -John disks.

Theorem D [69, Theorem 4.1]. *A simply connected domain D is a b -John disk if and only if there exists a constant $c \geq 1$ such that*

$$h_D(z_1, z_2) \leq c j'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Here the constants b and c depend only on each other.

On the other hand, Broch in his Ph.D thesis characterized John disks in terms of a bound for hyperbolic distance h_D with an additive constant.

Theorem E [25, Theorem 6.2.9]. *A simply connected (Jordan) domain D is a b -John disk if and only if there are constants c and d such that*

$$h_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2) + d$$

for all pairs of $z_1, z_2 \in D$, where c and d depend only on b , and b depends only on c and d .

In view of a comparison with Theorem B, Broch raised the following.

Conjecture F [25, Conjecture 6.2.12]. *A simply connected (Jordan) domain D in \mathbb{C} is a b -John disk if and only if there is a constant c such that*

$$h_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Here the constants b and c depend only on each other.

Recently, in [121], Wang, Huang, Ponnusamy and Chu proved that the sufficiency part in Conjecture F is actually true.

Theorem G [121, Corollary 2.3]. *Suppose that D is simply connected and that there is a constant c such that*

$$h_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Then D is a b -John disk with $b = b(c)$.

In addition to the above result, the authors in the same article [121] constructed two examples, one for a bounded John disk and the other for an unbounded John disk, and showed that the necessity in Conjecture F fails to hold. In view of this development and the importance of these domains in function theory, the following question is natural.

Problem 3.3. Is it true that D is a uniform domain if and only if there is a constant c such that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$.

We also collect a number of relevant results for completeness.

Theorem H. *Let D be a simply connected domain. Then for all $z_1, z_2 \in D$ we have*

$$\begin{aligned} \frac{1}{2} k_D(z_1, z_2) &\leq h_D(z_1, z_2) \leq 2 k_D(z_1, z_2), \\ k_D(z_1, z_2) &\geq \log \left(1 + \frac{|z_1 - z_2|}{\text{dist}(z_j, \partial D)} \right) \quad (j = 1, 2) \end{aligned}$$

and

$$(3.6) \quad j_D(z_1, z_2) \leq 2k_D(z_1, z_2).$$

Note that the first two inequalities in Theorem H are due to Gehring and Osgood [40] while the inequality (3.6) may be obtained, for instance, from [41, Lemma 2.1] and [120, Exercise 2.40] (see also [119] and [49, Section 5]). In fact, the inequality (3.6) holds for all proper subdomains D of \mathbb{R}^n . On the other hand, concerning K -quasi-isotropic domains (see Chapter 2 for the definition), Hästö proved the following analogous result.

Theorem I [49, Corollary 5.4]. *If a domain $D \subsetneq \mathbb{R}^n$ is K -quasi-isotropic, then*

$$k_D(z_1, z_2)/K \leq \tilde{a}_D(z_1, z_2) \leq 2k_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, where the second inequality always holds.

Theorem J [52, Corollary 5.4]. *For $D \subsetneq \mathbb{R}^n$ the following are equivalent:*

1. D is quasi-isotropic;
2. $\tilde{a}_D \approx k_D$; and
3. $j_D \lesssim \tilde{a}_D$.

Here is a simple result which illustrates the usefulness of our investigation and the proof of it is a consequence of Theorems D, H and I.

Theorem 3.4. *Let D be a simply connected domain. If D is a K -quasi-isotropic domain and there exists a constant c such that*

$$\tilde{a}_D(z_1, z_2) \leq c j'_D(z_1, z_2)$$

for all z_1 and z_2 in D , then D is a b -John disk, where b depends only on c and K .

Theorem 3.4 stimulates us to discuss the relation between K -quasi-isotropic domains and John domains. Then we ask the following.

Problem 3.5. Suppose that D is a simply connected domain. Is it true that D is a K -quasi-isotropic domain if and only if D is a b -John disk, where constants K and b depend only on each other?

The main aim of this chapter is to discuss Problems 3.3 and 3.5 which will be presented in the following sections.

3.2. Uniformity and λ -Apollonian Metric

In this section we present complete solution to Problem 3.3. We show that the necessary part is true in simply connected domains. More precisely, we give a number of examples in which the necessary as well as the sufficiency parts are not true in general.

There exist a number of alternative characterizations of uniform domains. However, it is a non-trivial task to verify whether a given domain is uniform. In [83, Examples 2.4(1)] Näkki and Väisälä stated that the exterior of a ball is a John domain. Although it seems from the definition that the exterior of a ball is uniform, because of independent interest, we present a proof below.

Lemma 3.6. *The domain $D = \mathbb{C} \setminus \overline{\mathbb{D}}$ is uniform.*

Proof. Let $z_1, z_2 \in D$, and recall the notation $\delta(z) = \text{dist}(z, \partial D)$. Without loss of generality we assume that $\delta(z_1) = \min_{j=1,2} \delta(z_j)$.

If $\delta(z_1) \geq \frac{1}{4}$, then we pick $z'_2 \in [0, z_2]$ such that $\delta(z'_2) = \delta(z_1)$. Therefore, z_1 and z'_2 divide the circle $S(0, |z_1|) = \{z \in \mathbb{C} : |z| = |z_1|\}$ into two parts: γ_1 and γ'_1 with $\ell(\gamma_1) \leq \ell(\gamma'_1)$. Define $\gamma = \gamma_1 \cup [z'_2, z_2]$. Then we have

$$\ell(\gamma) \leq \frac{\pi}{2}|z_1 - z'_2| + |z_2 - z'_2| \leq \frac{\pi + 2}{2}|z_1 - z_2|.$$

Given a $z \in \gamma$, if $z \in \gamma_1$, then we have

$$\ell(\gamma_1[z_1, z]) \leq \pi(\delta(z) + 1) \leq 5\pi\delta(z).$$

If $z \in [z'_2, z_2]$, then

$$\ell(\gamma_1) + \ell([z'_2, z]) \leq 5\pi\delta(z_1) + \delta(z) - \delta(z'_2) \leq 5\pi\delta(z).$$

Consequently, for each $z \in \gamma$, we have

$$\min_{j=1,2} \ell(\gamma[z_j, z]) \leq 5\pi\delta(z).$$

Now we assume that $\delta(z_1) < \frac{1}{4}$. We need to examine two cases.

Case I: Let $\delta(z_2) \geq \frac{1}{2}$.

Consider the half line L starting from the origin O and passing through z_1 , and let $z'_1 \in L$ with $\delta(z'_1) = \delta(z_2)$. Then z'_1 and z_2 divide the circle $S(0, \delta(z_2))$ into two parts: γ_2 and γ'_2 with $\ell(\gamma_2) \leq \ell(\gamma'_2)$.

Now, we let $\gamma = \gamma_2 \cup [z_1, z'_1]$. First notice that

$$\begin{aligned}\ell(\gamma) &\leq |z_1 - z'_1| + \ell(\gamma_2) \\ &\leq \delta(z_2) + \pi(1 + \delta(z_2)) \\ &\leq 2(3\pi + 1)|z_1 - z_2|,\end{aligned}$$

because $|z_1 - z_2| \geq |z_2| - |z_1| \geq \delta(z_2) - \frac{1}{4} \geq \frac{1}{2}\delta(z_2)$. For $z \in \gamma$, if $z \in [z_1, z'_1]$, we find that

$$\ell([z_1, z]) \leq \delta(z).$$

If $z \in \gamma_2$, we see that

$$\begin{aligned}\ell(\gamma_2(z_2, z)) &\leq \ell(\gamma_2) \leq \frac{\pi}{2}|z'_1 - z_2| \\ &\leq \frac{\pi}{2}(|z_2| + |z'_1|) = \pi(\delta(z_2) + 1) \\ &\leq 3\pi\delta(z)\end{aligned}$$

and so, for each $z \in \gamma$,

$$\min_{j=1,2} \ell(\gamma[z_j, z]) \leq 3\pi\delta(z).$$

Case II: Let $\delta(z_2) < \frac{1}{2}$.

Subcase I: First we consider the range $\frac{\pi}{18} \leq \angle z_1 O z_2 \leq \pi$.

Let L_1 be the half line starting from O and passing through z_1 , and let L_2 be the half line starting from O and passing through z_2 . Choose $z'_1 \in L_1$ and $z'_2 \in L_2$ with $\delta(z'_1) = \delta(z'_2) = \frac{1}{2}$. Then z'_1 and z'_2 divide the circle $S(0, \frac{3}{2})$ into two parts: γ_3 and γ'_3 with $\ell(\gamma_3) \leq \ell(\gamma'_3)$. Let $\gamma = [z_1, z'_1] \cup \gamma_3 \cup [z'_2, z_2]$. As in the previous case, this yields that

$$\begin{aligned}\ell(\gamma) &\leq \frac{1}{2} + \frac{1}{2} + \ell(\gamma_3) \leq 1 + \frac{3\pi}{2} \\ &\leq 2\pi \frac{\sin \frac{17\pi}{36}}{\sin \frac{\pi}{18}} |z_2 - z_1|.\end{aligned}$$

Now, for $z \in \gamma$, if $z \in [z_j, z'_j]$ ($j = 1, 2$), we then have

$$\ell([z_j, z]) \leq \delta(z).$$

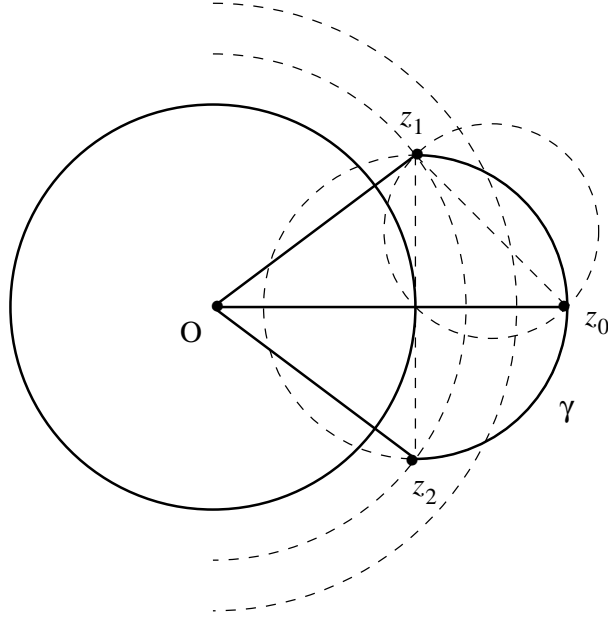


FIGURE 3.1. The domain $D = \mathbb{C} \setminus \overline{\mathbb{D}}$.

On the other hand, for $z \in \gamma_3$, we have

$$\begin{aligned} \ell(\gamma(z_1, z)) &\leq |z'_1| - |z_1| + \ell(\gamma_3) \leq \frac{1}{2} + \pi \left(1 + \frac{1}{2}\right) \\ &\leq (3\pi + 1)\delta(z), \end{aligned}$$

since $\delta(z) = \frac{1}{2}$. The above observations imply that

$$\min_{j=1,2} \ell(\gamma[z, z_j]) \leq (3\pi + 1)\delta(z)$$

for each $z \in \gamma$.

Subcase II: Consider the case $\angle z_1 O z_2 < \frac{\pi}{18}$.

Let γ be the half circle of $S\left(\frac{z_1+z_2}{2}, \frac{|z_1-z_2|}{2}\right)$ divided by z_1 and z_2 , which satisfies the condition $\angle O z_1 z_0 > \frac{\pi}{2}$, where $z_0 \in \gamma$ with $\ell(\gamma[z_1, z_0]) = \ell(\gamma[z_2, z_0])$, see Figure 3.1.

Clearly,

$$\ell(\gamma) \leq \frac{\pi}{2}|z_1 - z_2|.$$

Now, we claim that

$$\min_{j=1,2} \ell(\gamma[z_j, z]) \leq \pi\delta(z)$$

for $z \in \gamma$. To establish the claim, we first observe that for any $z \in \gamma[z_1, z_0]$, $\angle zz_1O \geq \frac{13\pi}{18}$. Hence for such z , we obtain that

$$\begin{aligned} |z| &= \sqrt{|z_1|^2 + |z_1 - z|^2 - 2|z_1||z_1 - z| \cos \angle Oz_1z} \\ &\geq \sqrt{1 + |z_1 - z|^2 + |z_1 - z|}. \end{aligned}$$

Since

$$\delta(z) = |z| - 1 \geq \sqrt{1 + |z_1 - z|^2 + |z_1 - z|} - 1$$

and

$$|z_1 - z| \leq 2 \left(\sqrt{1 + |z_1 - z|^2 + |z_1 - z|} - 1 \right) \leq 2\delta(z),$$

we deduce that

$$\ell(\gamma[z_1, z]) \leq \frac{\pi}{2}|z_1 - z| \leq \pi\delta(z).$$

On the other hand, if $z \in \gamma[z_0, z_2]$, we can find $z' \in \gamma[z_1, z_0]$ with $\ell(\gamma[z_1, z']) = \ell(\gamma[z, z_2])$ which shows that

$$\ell(\gamma[z, z_2]) = \ell(\gamma[z_1, z']) \leq \pi\delta(z') \leq \pi\delta(z).$$

The last two inequalities complete the proof of our claim. \square

We also need the following result from [80] which says that quasiconformal images of uniform domains are uniform.

Lemma 3.7. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a quasiconformal mapping and $D \subset \mathbb{R}^n$ be a uniform domain. Then $f(D)$ is a uniform domain.*

Theorem 3.8. *Suppose D is a simply connected domain. If D is uniform, then there exists a constant c such that*

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$.

Proof. Recall that a simply connected subdomain of the plane is uniform if and only if it is a quasidisk, and in general a uniform domain is a quasicircle domain, cf. [37]. Since D is given to be uniform, it is a quasidisk. Further, Theorems B, H and Lemma 3.1 imply that

$$j_D(z_1, z_2) \leq 4h_D(z_1, z_2) \leq 4c' \alpha_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, where $c = 4c'$ is a constant. \square

It is natural to ask whether Theorem 3.8 holds for multiply connected domains. The following example shows that the answer is negative.

Example 3.9. Let $D = \mathbb{C} \setminus D_1$ where $D_1 = \{x + iy : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$. Then D is a uniform domain, but there does not exist any constant c such that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$.

Solution. At first, we prove that $D = \mathbb{C} \setminus D_1$ is uniform, where $D_1 = \{x + iy : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$. Since D_1 is a quasidisk, there exists a K -quasiconformal mapping $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $D_1 = f(\mathbb{D})$. By Lemmas 3.6 and 3.7, $D = \mathbb{C} \setminus D_1$ is a uniform domain.

Next we prove that there does not exist any constant c such that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$.

Now $\partial D = L_1 \cup L_2 \cup L_3 \cup L_4$, where

$$\begin{aligned} L_1 &= \{x + iy : x = -\frac{1}{2}, |y| \leq \frac{1}{2}\}, \\ L_2 &= \{x + iy : y = -\frac{1}{2}, |x| \leq \frac{1}{2}\}, \\ L_3 &= \{x + iy : x = \frac{1}{2}, |y| \leq \frac{1}{2}\}, \\ L_4 &= \{x + iy : y = \frac{1}{2}, |x| \leq \frac{1}{2}\}, \end{aligned}$$

see Figure 3.2. Then $\partial D = \cup_{j=1}^4 L_j$. For $x \leq -\frac{3}{2}$, consider the two points $z_1 = x + \frac{1}{2}i$ and $z_2 = x - \frac{1}{2}i$ in D . Clearly $\delta(z_2) \geq 1$.

We will prove the inequalities

$$(3.7) \quad \frac{\lambda_D(z_1, a)}{\lambda_D(z_2, a)} \leq \sqrt{1 + \frac{1}{\delta(z_2)^2}}$$

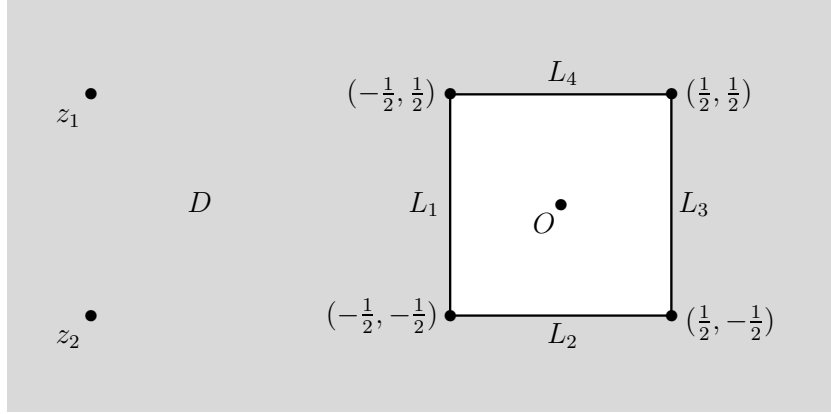


FIGURE 3.2. D , the exterior of the region bounded by L_1, L_2, L_3, L_4 .

and

$$(3.8) \quad \frac{\lambda_D(z_2, b)}{\lambda_D(z_1, b)} \leq \sqrt{1 + \frac{1}{\delta(z_1)^2}}$$

for all $a, b \in \partial_r D$. Obviously it suffices to prove the inequality (3.7), as the proof of the inequality (3.8) follows by symmetry.

We begin by observing that for the case $a \in L_1$, we have

$$\frac{\lambda_D(z_1, a)}{\lambda_D(z_2, a)} \leq \frac{\sqrt{|z_1 - z_2|^2 + \delta(z_2)^2}}{\delta(z_2)} = \sqrt{1 + \frac{1}{\delta(z_2)^2}}$$

since

$$\lambda_D(z_1, a) \leq |z_1 + \frac{1}{2} + \frac{i}{2}| = \sqrt{|z_1 - z_2|^2 + \delta(z_2)^2} \quad \text{and} \quad \lambda_D(z_2, a) \geq \delta(z_2).$$

Secondly, for the case $a \in L_2$, we find that

$$\frac{\lambda_D(z_1, a)}{\lambda_D(z_2, a)} = \frac{\sqrt{1 + \delta(z_2)^2} + |a + \frac{1}{2} + \frac{i}{2}|}{\delta(z_2) + |a + \frac{1}{2} + \frac{i}{2}|} \leq \sqrt{1 + \frac{1}{\delta(z_2)^2}}.$$

Thirdly, for the case $a \in L_4$, we easily obtain that

$$\frac{\lambda_D(z_1, a)}{\lambda_D(z_2, a)} = \frac{\delta(z_1) + |a + \frac{1}{2} - \frac{i}{2}|}{\sqrt{1 + \delta(z_2)^2} + |a + \frac{1}{2} - \frac{i}{2}|} \leq \sqrt{1 + \frac{1}{\delta(z_2)^2}}.$$

Finally, for the last case $a \in L_3$, we see that there exists a point $p = \frac{1}{2} + is \in L_3$ with $-\frac{1}{2} < s < 0$, such that

$$1 + \delta(z_2) + \left| p - \frac{1}{2} - \frac{i}{2} \right| = 1 + \sqrt{1 + \delta(z_2)^2} + \left| p - \frac{1}{2} + \frac{i}{2} \right|.$$

Indeed, for any such point we have $s = (\delta(z_2) - \sqrt{1 + \delta(z_2)^2})/2$. Similarly, one can see that there exists a point $q = \frac{1}{2} + it$ with $0 < t < \frac{1}{2}$, such that

$$1 + \delta(z_2) + \left| q - \frac{1}{2} + \frac{i}{2} \right| = 1 + \sqrt{1 + \delta(z_2)^2} + \left| q - \frac{1}{2} - \frac{i}{2} \right|.$$

Now, if the point a lies below the point p then by the same argument as in the second case, we obtain (3.7). If the point a lies above the point q , then arguing as in the third case verifies the inequality (3.7). If the point a lies between the points p and q , then it follows that there exists a point $z = \frac{1}{2} + ir$ ($-\frac{1}{2} \leq r < s$) such that

$$(3.9) \quad \frac{1 + \delta(z_2) + |z - \frac{1}{2} - \frac{i}{2}|}{1 + \delta(z_2) + |z - \frac{1}{2} + \frac{i}{2}|} = \sqrt{1 + \frac{1}{\delta(z_2)^2}}.$$

In fact, it is easy to see that

$$r = \frac{(\delta(z_2) - \sqrt{1 + \delta(z_2)^2})(\delta(z_2) + \frac{3}{2})}{\delta(z_2) + \sqrt{1 + \delta(z_2)^2}} \geq -\frac{1}{2},$$

since $\delta(z_2) \geq 1$. Consequently,

$$\begin{aligned} \frac{\lambda_D(z_1, a)}{\lambda_D(z_2, a)} &= \frac{1 + \delta(z_2) - \operatorname{Im}(a) + \frac{1}{2}}{1 + \delta(z_2) + \operatorname{Im}(a) + \frac{1}{2}} \\ &\leq \frac{1 + \delta(z_2) - s + \frac{1}{2}}{1 + \delta(z_2) + s + \frac{1}{2}} \\ &\leq \frac{1 + \delta(z_2) - r + \frac{1}{2}}{1 + \delta(z_2) + r + \frac{1}{2}} \\ &= \sqrt{1 + \frac{1}{\delta(z_2)^2}}, \end{aligned}$$

where the last equality occurs by (3.9). The proof of (3.7) is completed. Combining (3.7) and (3.8) gives that

$$\begin{aligned} \alpha'_D(z_1, z_2) &= \sup_{w_1, w_2 \in \partial_r D} \log \frac{\lambda_D(z_1, w_1) \lambda_D(z_2, w_2)}{\lambda_D(z_1, w_2) \lambda_D(z_2, w_1)} \\ &\leq \log \left(1 + \frac{1}{\delta(z_1)^2} \right), \end{aligned}$$

since $\delta(z_1) = \delta(z_2)$.

Finally, suppose on the contrary that there exists a constant c such that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Denote $y = \frac{1}{\delta(z_1)}$. Then, on one hand, we have

$$2 \log(1 + y) \leq c \log(1 + y^2).$$

But, on the other hand, we see that

$$\lim_{y \rightarrow 0} \frac{\log(1 + y)^2}{\log(1 + y^2)} = \infty.$$

This contradiction completes the solution. \square

As a consequence of the following example, we conclude that the converse part of Theorem 3.8 is not true in general.

Example 3.10. Let $D = \{x + iy : x > 0, |y| < 1\}$. Then there exists a constant c such that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, but D is not a uniform domain.

Solution. Theorem C implies that D is not a John disk and hence it is not uniform. Since D is convex, we have $j_D \leq 2\alpha_D$, by [113, Theorem 4.2]. Thus by Lemma 3.1 we have $j_D(z_1, z_2) \leq 2\alpha'_D(z_1, z_2)$ for all $z_1, z_2 \in D$. \square

In the following, we construct an example and show that the converse part of Theorem 3.8 does not hold in the case of D being multiply connected domains.

Example 3.11. Let $r = \frac{2 \tan(\pi/36)}{1 + 2 \tan(\pi/36)}$ and $D = D_1 \setminus D_2$, where $D_1 = \{x + iy : x > 0, |y| < 1\}$ and $D_2 = \{x + iy : r \leq x \leq 2 - r, |y| \leq 1 - r\}$. Then there exists a constant c such that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$, but D is not a uniform domain.

Solution. For our solution, we need the following lemma whose proof is easy to obtain by using basic trigonometry and so we omit the details.

Lemma 3.12. *Let D be the same as that in Example 3.11 and $x_1 = r + (1-r)i$, $y_1 = r + i$, $x_2 = 2 - r + (1-r)i$. Then $x_1, y_1, x_2 \in \partial D$ and $\angle x_1 x_2 y_1 = \frac{\pi}{36}$.*

Now, let z_1, z_2 be any two points in D . Without loss of generality we may assume that $\delta(z_1) = \min_{j=1,2} \delta(z_j)$, where $\delta(z_j) = \text{dist}(z_j, \partial D)$. Let $z \in \partial D$ be such that $\delta(z_1) = |z_1 - z|$.

If $|z_1 - z_2| \geq 3\delta(z_1)$, then

$$\begin{aligned} \alpha_D(z_1, z_2) &\geq \log \frac{|z - z_2|}{|z - z_1|} \\ &\geq \log \left(1 + \frac{|z_1 - z_2|}{3\delta(z_1)} \right). \end{aligned}$$

By Lemmas 3.1 and 3.2, we have

$$\begin{aligned} j_D(z_1, z_2) &\leq 2 \log \left(1 + \frac{|z_1 - z_2|}{\delta(z_1)} \right) \\ &\leq 6 \log \left(1 + \frac{|z_1 - z_2|}{3\delta(z_1)} \right) \\ &\leq 6 \alpha_D(z_1, z_2) \\ (3.10) \quad &\leq 6 \alpha'_D(z_1, z_2). \end{aligned}$$

If $|z_1 - z_2| < 3\delta(z_1)$ and $z \in \partial D_1$, then, by Theorem 4.2 in [113] and Lemma 3.1, it follows that

$$(3.11) \quad j_D(z_1, z_2) \leq 2 \log \left(1 + \frac{|z_1 - z_2|}{\delta(z_1)} \right) \leq 2a'_{D_1}(z_1, z_2) \leq 2\alpha'_D(z_1, z_2).$$

In the following, we always assume that $|z_1 - z_2| < 3\delta(z_1)$ and $z \in \partial D_2$.

Following the notation of Lemma 3.12, let $x_1 = r + (1-r)i$, $x_2 = 2 - r + (1-r)i$, $x_3 = 2 - r + (r-1)i$ and $x_4 = r + (r-1)i$, see Figure 3.3.

Case I: First we consider the case $z = x_1$.

Without loss of generality, we may assume $z_1 \in W_1$, where W_1 denotes the closure of the triangular domain with vertices $r + (1-r)i$, $r + i$ and i . Then

$$z_2 \in W_1 \cup W_2 \cup W_3 \cup W_4,$$

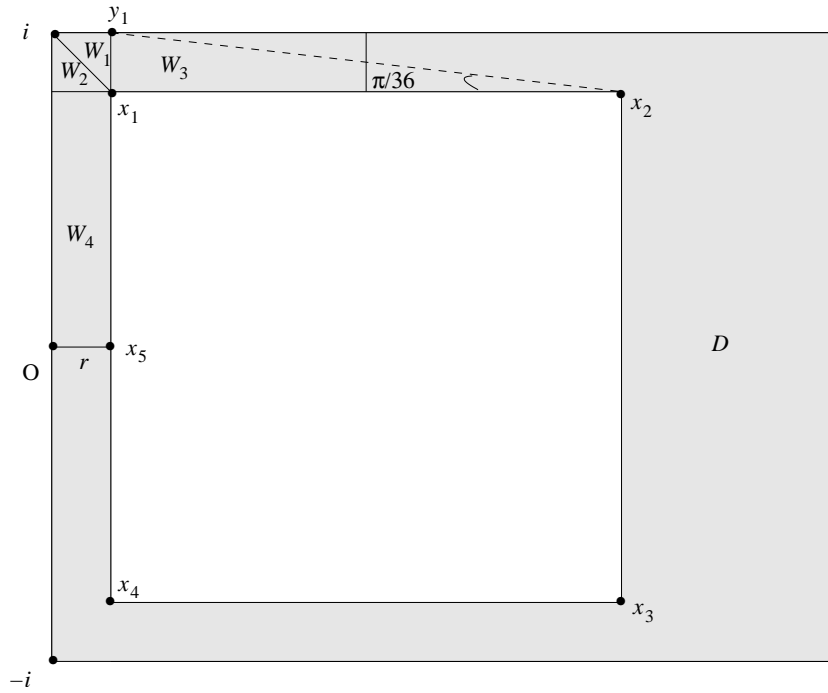


FIGURE 3.3. D , the domain exterior of the rectangle (with vertices x_1, x_2, x_3, x_4) and contained in the half strip.

where W_2 denotes the closure of the triangular domain with the vertices $r + (1 - r)i$, i and $(1 - r)i$; W_3 the closure of the rectangular domain with the vertices $r + (1 - r)i$, $1 + (1 - r)i$, $1 + i$ and $r + i$; and W_4 the closure of the rectangular domain with the vertices 0 , r , $r + (1 - r)i$ and $(1 - r)i$, see Figure 3.3. We divide our discussions into two subcases.

Subcase I: The subcase $z_2 \in W_1$.

In the following, when we mention an angle we always mean that one which is not greater than π .

Obviously, $\angle z_2 z_1 z \geq \frac{3\pi}{8}$. Next we obtain that, if $\angle z_2 z_1 z \geq \frac{19\pi}{36}$, then

$$\frac{|z - z_2|}{|z - z_1|} \geq \frac{\sqrt{|z - z_1|^2 + |z_2 - z_1|^2 + 2c_0|z - z_1||z_2 - z_1|}}{|z - z_1|} \geq \sqrt{1 + \frac{2c_0|z_2 - z_1|}{\delta(z_1)}},$$

where $c_0 = \sin \frac{\pi}{36}$, which yields that

$$\alpha_D(z_1, z_2) \geq \log \frac{|z - z_2|}{|z - z_1|} \geq \log \sqrt{1 + \frac{2c_0|z_2 - z_1|}{\delta(z_1)}}.$$

Appealing to Lemmas 3.1 and 3.2, we obtain

$$(3.12) \quad j_D(z_1, z_2) \leq 2 \log \left(1 + \frac{|z_1 - z_2|}{\delta(z_1)} \right) \leq \frac{2}{c_0} \alpha'_D(z_1, z_2).$$

If $\angle z_2 z_1 z \leq \frac{19\pi}{36}$, then, by Lemma 3.1, there must exist a point $a \in [x_1, x_2]$ or $[x_1, x_5]$ such that

$$\angle z_2 z_1 a = \frac{19\pi}{36}.$$

Elementary computations show that there exists a constant c_1 such that

$$\frac{|a - z_2|}{|a - z_1|} \geq \sqrt{1 + \frac{2c_0|z_1 - z_2|}{|a - z_1|}}$$

and

$$\alpha_D(z_1, z_2) \geq \log \sqrt{1 + \frac{c_1|z_1 - z_2|}{\delta(z_1)}},$$

where we can take

$$c_1 = \min \left\{ c_0, \frac{\sin \frac{\pi}{36} \sin \frac{\pi}{12}}{1 + 2 \sin \frac{\pi}{12}}, \frac{2 \sin \frac{\pi}{36} \sin \frac{\pi}{18}}{2 \sin \frac{\pi}{36} + \sin \frac{\pi}{18}} \right\} = \frac{\sin \frac{\pi}{36} \sin \frac{\pi}{12}}{1 + 2 \sin \frac{\pi}{12}}.$$

Therefore,

$$(3.13) \quad j_D(z_1, z_2) \leq 2 \log \left(1 + \frac{|z_1 - z_2|}{\delta(z_1)} \right) \leq \frac{4}{c_1} \alpha'_D(z_1, z_2).$$

Subcase II: The subcase $z_2 \in W_2 \cup W_3 \cup W_4$.

Arguing as in *Subcase I*, we see that there exists a constant c_2 such that

$$(3.14) \quad j_D(z_1, z_2) \leq \frac{4}{c_2} \alpha'_D(z_1, z_2),$$

where we can take

$$\begin{aligned} c_2 &= \min \left\{ c_0, \frac{2 \sin \frac{\pi}{36} \sin \frac{\pi}{18}}{\sin \frac{\pi}{18} + \sin \frac{7\pi}{36}}, \frac{2 \sin \frac{\pi}{36} \sin \frac{\pi}{18}}{\sin \frac{\pi}{18} + \sin \frac{5\pi}{18}}, \frac{\sin \frac{\pi}{36} \sin \frac{7\pi}{36}}{2(\sin \frac{7\pi}{36} + \sin \frac{11\pi}{36})}, \frac{\sin \frac{\pi}{36} \sin \frac{5\pi}{36}}{2(\sin \frac{5\pi}{36} + \sin \frac{13\pi}{36})} \right\} \\ &= \frac{\sin \frac{\pi}{36} \sin \frac{5\pi}{36}}{2(\sin \frac{5\pi}{36} + \sin \frac{13\pi}{36})}. \end{aligned}$$

Case II: The case $z \in (x_1, x_4]$.

Let $x_5 = r$. Without loss of generality, we may assume that $z \in (x_1, x_5]$. Clearly $z_1 \in W_4$. Then we see that

$$z_2 \in U_1 \cup W_3 \cup U_2,$$

where U_1 denotes the closure of the rectangular domain with the vertices $\operatorname{Im}(z_1)i$, $r + \operatorname{Im}(z_1)i$, $r+i$ and i ; and U_2 the closure of the rectangular domain with the vertices $(r-1)i$, $r + (r-1)i$, $r + \operatorname{Im}(z_1)i$ and $\operatorname{Im}(z_1)i$.

Subcase III: The subcase $z_2 \in U_1$.

First we observe that $\angle z_2 z_1 z > \frac{\pi}{4}$. If $\angle z_2 z_1 z \geq \frac{19\pi}{36}$, then

$$(3.15) \quad j_D(z_1, z_2) \leq \frac{2}{c_0} \alpha'_D(z_1, z_2).$$

On the other hand, if $\angle z_2 z_1 z < \frac{19\pi}{36}$, then there must exist a point $a \in [x_1, x_4]$ such that

$$\angle z_2 z_1 a = \frac{19\pi}{36}.$$

Obviously $\angle z z_1 a \leq \frac{5\pi}{18}$ and $\angle z_1 a z \geq \frac{2\pi}{9}$. Hence,

$$\frac{|a - z_2|}{|a - z_1|} \geq \sqrt{1 + \frac{2c_0|z_1 - z_2|}{|a - z_1|}}$$

and

$$\alpha_D(z_1, z_2) \geq \log \sqrt{1 + \frac{c_3|z_1 - z_2|}{\delta(z_1)}},$$

where c_3 can be taken as

$$c_3 = \frac{2 \sin \frac{\pi}{36} \sin \frac{2\pi}{9}}{\sin \frac{5\pi}{18} + \sin \frac{2\pi}{9}}.$$

Consequently,

$$(3.16) \quad j_D(z_1, z_2) \leq \frac{4}{c_3} \alpha'_D(z_1, z_2).$$

Subcase IV: The subcase $z_2 \in W_3$.

Our choice of point ensures that $\angle z_2 z_1 z \geq \frac{\pi}{4}$ or $\angle z_1 z_2 b \geq \frac{\pi}{4}$, where

$$b = \operatorname{Re}(z_2) + (1 - r)i.$$

Then there must exist a point $a \in [x_1, x_4]$ or $a \in [x_1, x_2]$ such that

$$\angle z_2 z_1 a = \frac{19\pi}{36} \quad \text{or} \quad \angle z_1 z_2 a = \frac{19\pi}{36}.$$

We have known that there exists a constant c_4 such that

$$(3.17) \quad j_D(z_1, z_2) \leq \frac{4}{c_4} \alpha'_D(z_1, z_2),$$

where we can take

$$c_4 = \frac{\sin \frac{\pi}{36} \sin \frac{2\pi}{9}}{2(\sin \frac{2\pi}{9} + \sin \frac{5\pi}{18})}.$$

Subcase V: The subcase $z_2 \in U_2$.

If $\angle z_2 z_1 z \geq \frac{19\pi}{36}$, then

$$\frac{|z - z_2|}{|z - z_1|} \geq \sqrt{1 + \frac{2c_0|z_1 - z_2|}{\delta(z_1)}}$$

and

$$(3.18) \quad j_D(z_1, z_2) \leq \frac{2}{c_0} \alpha'_D(z_1, z_2).$$

If $\angle z_2 z_1 z \leq \frac{19\pi}{36}$, then $\frac{17\pi}{36} \leq \angle z_1 z_2 z \leq \frac{\pi}{2}$ and there must exist a point $a \in [x_1, x_4]$ such that

$$\angle z_1 z_2 a = \frac{19\pi}{36}.$$

Therefore, there exists a constant c_5 such that

$$(3.19) \quad j_D(z_1, z_2) \leq \frac{4}{c_5} \alpha'_D(z_1, z_2),$$

where we can take

$$c_5 = \min \left\{ c_0, \frac{\sin \frac{\pi}{36} \sin \frac{17\pi}{36}}{2(\sin \frac{17\pi}{36} + \sin \frac{\pi}{36})} \right\} = \frac{\sin \frac{\pi}{36} \sin \frac{17\pi}{36}}{2(\sin \frac{17\pi}{36} + \sin \frac{\pi}{36})}.$$

By the symmetry of D , there is only one possibility about the place of z which needs to be discussed, which is:

Case III: The case $z \in [x_2, x_3]$.

By the above discussions, we may assume that $\delta(z_1) \geq r$. Then $z_1, z_2 \in W_5$, where W_5 denotes the closure of the rectangular domain with the vertices $2 + (r-1)i$, $3 - r + (r-1)i$, $3 - r + (1-r)i$ and $2 + (1-r)i$.

If $\angle z_2 z_1 z \geq \frac{19\pi}{36}$, then

$$(3.20) \quad j_D(z_1, z_2) \leq \frac{2}{c_0} \alpha'_D(z_1, z_2).$$

If $\angle z_2 z_1 z < \frac{19\pi}{36}$, then there exists a point $a \in [x_2, x_3]$ such that

$$\angle z_2 z_1 a = \frac{19\pi}{36} \text{ or } \angle z_1 z_2 a = \frac{19\pi}{36}.$$

We have known that there is a constant c_6 such that

$$(3.21) \quad j_D(z_1, z_2) \leq \frac{4}{c_6} \alpha'_D(z_1, z_2),$$

where we can take

$$c_6 = \min \left\{ \frac{2 \sin \frac{\pi}{36} \sin \frac{17\pi}{36}}{\sin \frac{17\pi}{36} + \sin \frac{\pi}{36}}, \frac{2 \sin \frac{\pi}{36} \sin \frac{4\pi}{9}}{\sin \frac{\pi}{18} + \sin \frac{4\pi}{9}} \right\} = \frac{2 \sin \frac{\pi}{36} \sin \frac{4\pi}{9}}{\sin \frac{\pi}{18} + \sin \frac{4\pi}{9}}.$$

Finally, we let

$$c = \max \left\{ 6, \frac{4}{c_i} : i = 0, \dots, 6 \right\}.$$

Then $c > 0$ and equations (3.10) – (3.21) show that

$$j_D(z_1, z_2) \leq c \alpha'_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. At last, the proof of D being not a uniform domain easily follows from the definition. \square

3.3. John Disks and Quasi-isotropic Domains

In this section, we give answer to Problem 3.5. We see that neither John disks are quasi-isotropic nor conversely.

From the definition of inner metric we have seen that $d_1 \approx d_2$ implies $\tilde{d}_1 \approx \tilde{d}_2$. Hence we have the following result which in view of Theorem J says that domains satisfying comparison property are quasi-isotropic.

Proposition 3.13. *For a domain $D \subsetneq \mathbb{R}^n$ if $\alpha_D \approx j_D$, then $\tilde{\alpha}_D \approx k_D$ holds.*

The following examples show that the necessary part in Problem 3.5 does not hold irrespective of whether D is bounded or unbounded.

Example 3.14. Let D be the domain as in Example 3.10. Then D is a quasi-isotropic domain, but D is not a John disk.

Solution. It follows from item (3) in Theorem C that D is not a John disk.

Since D is convex, by [113, Theorem 4.2], it follows that $j_D \leq 2\alpha_D$ and so, we have

$$j_D(z_1, z_2) \leq 2\alpha_D(z_1, z_2) \leq 2\tilde{a}_D(z_1, z_2)$$

for all $z_1, z_2 \in D$. Thus, Theorem J implies that D is quasi-isotropic. \square

Example 3.15. Let $D = D_1 \setminus D_2$, where $D_1 = \{x + iy : -1 < x < 0, -1 < y < 0\}$ and $D_2 = \{x + iy : x^2 + y^2 \leq 1, x < 0, y < 0\}$. Then D is quasi-isotropic, but not a John disk.

Solution. First we prove that D defined in Example 3.15 is quasi-isotropic. By Theorem J and Proposition 3.13, it suffices to prove the following:

$$(3.22) \quad \alpha_D \approx j_D.$$

It is well-known that $\alpha_D(z_1, z_2) \leq j_D(z_1, z_2)$ for all $z_1, z_2 \in D$. So, in order to prove (3.22), we only need to prove

$$(3.23) \quad j_D \lesssim \alpha_D.$$

For any $z_1, z_2 \in D$, without loss of generality, we may assume that $\delta(z_1) = \min_{j=1,2} \delta(z_j)$. Let $z \in \partial D$ be such that $\delta(z_1) = |z - z_1|$.

We need to deal with two cases.

Case I: The case $|z_1 - z_2| \geq 3\delta(z_1)$.

We see from the first part of the solution of Example 3.11 that

$$(3.24) \quad j_D(z_1, z_2) \leq 6\alpha_D(z_1, z_2).$$

Case II: The case $|z_1 - z_2| < 3\delta(z_1)$.

Define

$$\begin{aligned} L_1 &= \{x + iy : x = -1, -1 \leq y \leq 0\} \\ L_2 &= \{x + iy : y = -1, -1 \leq x \leq 0\} \\ L_3 &= \{x + iy : x^2 + y^2 = 1, x < 0, y < 0\}. \end{aligned}$$

If $z \in L_1 \cup L_2$, then $\angle z_2 z_1 z \geq \frac{\pi}{2}$ and there must exist a point $z_0 \in L_1 \cup L_2$ such that

$$\angle z_0 z_1 z = \frac{\pi}{6} \quad \text{and} \quad \frac{2\pi}{3} \leq \angle z_2 z_1 z_0 \leq \pi.$$

Then it follows that

$$\begin{aligned} \alpha_D(z_1, z_2) &\geq \log \frac{|z_2 - z_0|}{|z_1 - z_0|} \\ &= \log \frac{\sqrt{|z_1 - z_2|^2 + |z_1 - z_0|^2 - 2 \cos \angle z_2 z_1 z_0 |z_1 - z_2| |z_1 - z_0|}}{|z_1 - z_0|} \\ &\geq \log \sqrt{1 + \frac{\sqrt{3} |z_1 - z_2|}{2\delta(z_1)}}. \end{aligned}$$

Moreover, by Lemma 3.2, we get

$$(3.25) \quad j_D(z_1, z_2) \leq \frac{8}{\sqrt{3}} \alpha_D(z_1, z_2).$$

If $z \in L_3$, then $\angle z_2 z_1 z \geq \frac{\pi}{3}$ and there must exist a point $z_0 \in L_3$ such that $\angle z_2 z_1 z_0 = \frac{5\pi}{9}$.

It follows that there must exist c_1 such that

$$(3.26) \quad |z_1 - z_0| \leq c_1 \delta(z_1), \quad c_1 = 1 + \frac{\sin \frac{2\pi}{9}}{\sin \frac{\pi}{9}}.$$

Then (3.26) implies that

$$\begin{aligned} \alpha_D(z_1, z_2) &\geq \log \frac{|z_2 - z_0|}{|z_1 - z_0|} \\ &= \log \frac{\sqrt{|z_1 - z_2|^2 + 2 \sin \frac{\pi}{18} |z_1 - z_2| |z_1 - z_0|}}{|z_1 - z_0|} \\ &\geq \log \sqrt{1 + \frac{2 \sin \frac{\pi}{18} |z_1 - z_2|}{c_1 \delta(z_1)}}. \end{aligned}$$

On the other hand, Lemma 3.2 yields that

$$(3.27) \quad j_D(z_1, z_2) \leq 2 \log \left(1 + \frac{|z_1 - z_2|}{\delta(z_1)} \right) \leq c_2 \alpha_D(z_1, z_2), \quad c_2 = \frac{c_1}{\sin \frac{\pi}{18}}.$$

Therefore, using (3.24), (3.25) and (3.27), we obtain (3.23).

Now we prove that D is not a John disk. For this, we let

$$x_0 = -1, \quad x_1 = -1 - ti \in L_1 \quad \text{and} \quad x_2 = s - ti \in L_3 \quad (0 < t < 1/4).$$

Then $|x_1 - x_2| = |1 + s|$ and the straight crosscut $[x_1, x_2]$ divides the domain D into two subdomains which are denoted by D_1 and D_2 . Obviously,

$$\min_{j=1,2} \text{diam}(D_j) = |x_0 - x_2| = \sqrt{|1 + s|^2 + t^2} = \sqrt{2 + 2s}.$$

Suppose on the contrary that D is a John disk. Then Theorem C implies that there exists a constant c such that

$$\min_{j=1,2} \text{diam}(D_j) \leq c|x_1 - x_2|.$$

But then

$$\lim_{s \rightarrow -1} \frac{\sqrt{2 + 2s}}{|1 + s|} = \infty$$

which is a contradiction. This completes the solution. \square

Remark 3.16. By [49, Example 4.4] it follows that $\mathbb{H} \setminus [0, i]$ is not quasi-isotropic but is a John disk by Theorem C, where \mathbb{H} denotes the upper half-plane. As another motivation to Problem 3.5, we observe by a proof similar to [49, Example 4.4] that there exist bounded simply connected domains (eg. $\mathbb{D} \setminus [0, 1)$) which are John but are not quasi-isotropic. Also there exist doubly connected domains which are John domains, but not quasi-isotropic. For instance, [63, Example 3.11] gives that $\mathbb{D} \setminus \{0\}$ is not quasi-isotropic but is clearly a John domain.

Remark 3.17. Remark 3.16 shows that the sufficiency part in Problem 3.5 does not hold whether D is bounded or unbounded. These observations clearly provide us a solution to Problem 3.5. Examples 3.14 and 3.15 show that the inequality $\tilde{a}_D(z_1, z_2) \leq c j'_D(z_1, z_2)$ in Theorem 3.4 cannot be removed.

CHAPTER 4

ISOMETRIES OF SOME HYPERBOLIC-TYPE PATH METRICS

This chapter begins with a survey on isometries of some hyperbolic-type path metrics and continues with some old results as well as some new results. Section 4.1 deals with definitions and brief introduction to the isometry problems of such metrics. In Section 4.2, the work of Hästö [54] for the quasihyperbolic metric is described. In addition, we relate how Herron, Ibragimov and Minda [60] used circular geodesics and the curvature of the K–P metric to take care of its isometries in most domains. Finally, in Section 4.3 we show how the isometry problem can be solved for the K–P metric in doubly connected domains using a new concept which we call the hyperbolic medial axis, and we also present some new results for the quasihyperbolic metric and Ferrand’s metric.

The results of this chapter have been published in: **P. Hästö, Z. Ibragimov, D. Minda, S. Ponnusamy and S.K. Sahoo** Isometries of some hyperbolic type path metrics and the hyperbolic medial axis. *In the Tradition of Ahlfors-Bers, IV (Ann Arbor, MI, 2005)*, 63–74, *Contemp. Math.* **432**, Amer. Math. Soc., Providence, RI, 2007.

4.1. Introduction

A conformal path metric is a special kind of Finslerian metric, in which the density depends only on the location, not on the direction. If D is a connected subset of \mathbb{R}^n and p is a non-negative real valued function defined on D , then we can define such a metric by (1.2) with the density function $p(z)$. If p is a C^2 function, then we are in the standard Riemannian setting, but there is nothing preventing us from considering also a more general p .

In fact, choosing $p(z) = \delta(z)^{-1}$, where δ is the distance-to-the-boundary function, gives us the well-known quasihyperbolic metric. Despite the prominence of this metric, there have been almost no investigations of its geometry (some exceptions are [77, 79]). Part of the reason for this lack of geometrical investigations is probably that the density of the quasihyperbolic metric is not differentiable in the entire domain, which places the metric outside the standard framework of Riemannian metrics.

At least two modifications of the quasihyperbolic metric have been proposed that go some way to alleviate this problem. J. Ferrand [30] suggested replacing the density $\delta(z)^{-1}$ with $\sigma_D(x)$ defined by (1.4). Note that $\delta(x)^{-1} \leq \sigma_D(x) \leq 2\delta(x)^{-1}$, so the Ferrand metric and the quasihyperbolic metric are bilipschitz equivalent. Moreover, the Ferrand metric is Möbius invariant, whereas the quasihyperbolic metric is only Möbius quasi-invariant. A second variant was proposed more recently by R. Kulkarni and U. Pinkall [74], see also [61]. The K–P metric is defined by the density (1.5). This density satisfies the same estimates as Ferrand’s density, i.e. $\delta(x)^{-1} \leq \mu_D(x) \leq 2\delta(x)^{-1}$, and the K–P metric is also Möbius invariant. Although the Ferrand and the K–P metrics are in some sense better behaved than the quasihyperbolic metric, they suffer from the shortcoming that it is very difficult to get a grip on the density, even in simple domains.

Despite this, D. Herron, Z. Ibragimov and D. Minda [60] recently managed to solve the isometry problem for the K–P metric in most cases. Recall that by the isometry problem for the metric d we mean characterizing mappings $f : D \rightarrow \mathbb{R}^2$ which satisfy (1.1) for all $x, y \in D$. Notice that in some sense we are here dealing with two different metrics, due to the dependence on the domain. Hence the usual way of approaching the isometry problem is by looking at some intrinsic features of the metric which are then preserved under the isometry. Since irregularities (e.g. cusps) in the domain often lead to more distinctive features, this implies that the problem is often easier for more complicated domains. The work by Herron, Ibragimov and Minda [60] bears out this heuristic – they were able to show that all isometries of the K–P metric are Möbius mappings except possibly in simply and doubly connected domains. Their proof is based on studying the circular geodesics of the K–P metric. For the quasihyperbolic metric, formulas for the curvature were worked out in [79] (see Section 4.2) and were used in that paper to prove that all the isometries

of the disk are similarity mappings. The proofs in [54] are based on both the curvature and its gradient and work for domains with C^3 boundary.

Besides the aforementioned works, the proofs in this chapter were inspired by the work on the isometries of other (non-path) metrics [56, 57, 59].

Notation

We tacitly identify \mathbb{R}^2 with \mathbb{C} , and speak about real and imaginary axes, etc. By $B(x, r)$ we denote a disk with center x and radius r , and by $(x, y]$ or $[x, y)$ the half-open (or semi open) segment between x and y .

The cross-ratio $|a, b, c, d|$ is defined by

$$|a, b, c, d| = \frac{|a - c| |b - d|}{|a - b| |c - d|}$$

for distinct points $a, b, c, d \in \overline{\mathbb{R}^n}$, with the understanding that $|\infty - x|/|\infty - y| = 1$ for all $x, y \in \mathbb{R}^n$. A homeomorphism $f: \overline{\mathbb{R}^n} \rightarrow \overline{\mathbb{R}^n}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for every quadruple of distinct points $a, b, c, d \in \overline{\mathbb{R}^n}$. A mapping of a subdomain of $\overline{\mathbb{R}^n}$ is Möbius if it is the restriction of a Möbius mapping defined on $\overline{\mathbb{R}^n}$. For more information on Möbius mappings, see, for example, [7, Section 3]. Note that a Möbius mapping can always be decomposed as $i \circ s$, where i is an inversion or the identity and s is a similarity (i.e. a homeomorphism satisfying $|s(x) - s(y)| = c|x - y|$ for some positive constant c).

4.2. Isometries of k_D and μ_D ; Known Results

4.2.1. Isometries of the quasihyperbolic metric k_D

By $f \in C^k$ we mean that f is a k times continuously differentiable function. By a C^k domain we mean a domain whose boundary can be locally represented as the graph of a C^k function. To carry out step (2) of the isometry program discussed in Section 1.3, the following result was proved in [54, Proposition 2.2]:

Proposition 4.1. *Let $D \subsetneq \mathbb{R}^2$ be a C^1 domain, and let $f: D \rightarrow \mathbb{R}^2$ be a quasihyperbolic isometry which is also Möbius. If D is not a half-plane, then f is a similarity.*

Note that if we do not assume C^1 boundary, then there are some domains with non-similarity isometries: punctured planes $\mathbb{R}^2 \setminus \{a\}$ and sector domains (i.e. domains whose boundary consists of two rays). In both cases inversions centered at the distinguished boundary point (a or the vertex of the sector) are also isometries. The previous proposition strongly suggests that these are all the examples of domains with non-similarity isometries.

An immediate consequence is the solution of the isometry problem in higher dimensions for C^1 domains [54, Corollary 2.3]:

Corollary 4.2. *Let D be a C^1 domain in \mathbb{R}^n , $n \geq 3$, which is not a half-space. Then every quasihyperbolic isometry is a similarity mapping.*

The *medial axis* of D is the set of centers of maximal balls (with respect to the inclusion order) in D . The medial axis is denoted by $\text{MA}(D)$. For some mathematical investigations of the medial axis, see [26, 28], and for an application to the quasiworld see [15].

By R_ζ we denote the reciprocal of the curvature of ∂D at the boundary point ζ . The principal tool in [54] for attacking the main problem in the isometry program (namely, step (3)) is the following curvature formula based on the estimates of Martin and Osgood [79].

Proposition 4.3 (Proposition 3.2, [54]). *Let $D \subsetneq \mathbb{R}^2$ be a C^2 domain and $z \in D \setminus \text{MA}(D)$ have closest boundary point $\zeta \in \partial D$. Then*

$$\kappa_D(z) = -\frac{R_\zeta}{R_\zeta - \delta(z)} = -\frac{1}{1 - \delta(z)/R_\zeta}.$$

If z lies on the medial axis, then $\kappa_D(z) = -\infty$.

Using this proposition, the following theorem was proved in [54, Theorem 4.3].

Theorem 4.4. *Let $D \subsetneq \mathbb{R}^2$ be a C^3 domain that is not a half-plane. Then every isometry $f: D \rightarrow \mathbb{R}^2$ of the quasihyperbolic metric is a similarity mapping.*

The idea of the proof is the following: Let $z \in D \setminus \text{MA}(D)$ and let ζ_z be its unique nearest boundary point. Then the half-open segment $[z, \zeta_z)$ is a geodesic half-line with respect to the quasihyperbolic metric. The proof of the theorem is based on showing that this type of geodesic is somehow special and is thus mapped to another geodesic half-line of the same type. There are a couple of different cases based on the curvature R_ζ at the nearest boundary point, but essentially this part of the proof is based on Proposition 4.3. It is then shown that an isometry maps a segment to a segment, which implies that it is a Möbius mapping. The proof is concluded by applying Proposition 4.1, which says that the Möbius isometry is a similarity.

In fact, the smoothness assumption on the boundary of the domain can be dropped to C^2 , except in two special cases, namely, when the domain is strictly convex or strictly concave! Corollary 4.13, proved below, takes care of the concave case, so only the convex case remains.

4.2.2. Isometries of the K–P metric μ_D

Extremal disks and *circular geodesics* are important objects when discussing the isometries of the K–P metric. Consider a domain D in \mathbb{C} with $\text{card}(\partial D) \geq 2$. A disk or a half-plane $B \subset D$ with $\text{card}(\partial B \cap \partial D) \geq 2$ is called an extremal disk. We call Γ a circular geodesic in D if there exists an extremal disk $B \subset D$ such that Γ is a hyperbolic geodesic line in B with endpoints in $\partial B \cap \partial D$. While the definition may not suggest the importance of extremal disks, it is the existence of a unique extremal disk associated to each point in the domain that plays a crucial role in the study of the K–P metric.

More precisely, given a domain $D \subset \mathbb{C}$ with $\text{card}(\partial D) \geq 2$ and a point $z \in D$, let i_z be the inversion in a circle centered at z with radius 1. Then the complement of $i_z(D)$ is a compact set in \mathbb{C} and hence by Jung’s Theorem (see [12, 11.5.8, p. 357]) there exists a unique disk B of smallest radius whose closure contains the set. In particular, $\text{card}(\partial B \cap \partial D) \geq 2$ and $z \in i_z(\overline{\mathbb{C}} \setminus \overline{B}) \subset D$. Hence the set $i_z(\overline{\mathbb{C}} \setminus \overline{B})$ is an extremal disk, which is properly called the extremal disk at z and is denoted by B_z . An observant reader will notice that B_z is also the extremal disk for each point of a circular geodesic contained in B_z . Using more delicate arguments it is proved that B_z is the extremal disk

for each point of \hat{K}_z and only for these points, where \hat{K}_z is the hyperbolic convex hull of the set $\partial B_z \cap \partial D$ in B_z (see [60, Proposition 2.5]). In particular, for each pair of points $z, w \in D$, we have either $\hat{K}_z = \hat{K}_w$ or $\hat{K}_z \cap \hat{K}_w = \emptyset$.

Another important property of the extremal disks is that $\mu_D(z) = \lambda_{B_z}(z)$, where λ_{B_z} is the density of the hyperbolic metric in B_z . In particular,

$$\mu_D(z) = \sup_{a, b \in \partial B_z} \frac{|a - b|}{|a - z| |z - b|}$$

and if γ_z is any hyperbolic geodesic in B_z passing through z , then

$$\mu_D(z) = \frac{|a(z) - b(z)|}{|a(z) - z| |z - b(z)|},$$

where $a(z) \in \partial B_z$ and $b(z) \in \partial B_z$ are the endpoints of γ_z . Hence by the monotonicity of the Ferrand metric we obtain

$$\mu_D(z) = \sup_{a, b \in \partial B_z} \frac{|a - b|}{|a - z| |z - b|} = \sigma_{B_z}(z) \geq \sigma_D(z),$$

and $\mu_D(z) = \sigma_D(z)$ if and only if z lies on a circular geodesic. Using Jung's Theorem we also obtain that $\mu_D(z) \leq (2/\sqrt{3})\sigma_D(z)$ (see [60] for details).

We also need the following lower bound for the Ferrand (and hence for the K-P) distance. Given a domain $D \subset \overline{\mathbb{C}}$ with $\text{card}(\partial D) \geq 2$ and points $z, w \in D$, we consider the following distance function

$$s_D(z, w) = \log \left(1 + \sup_{a, b \in \partial D} \frac{|a - b| |z - w|}{|a - z| |b - w|} \right).$$

The function s_D , introduced by Seittenranta [113], defines a metric in D and since

$$\lim_{w \rightarrow z} \frac{s_D(z, w)}{|z - w|} = \sup_{a, b \in \partial D} \frac{|a - b|}{|a - z| |z - b|} = \sigma_D(z) \quad \text{for each } z \in D \cap \mathbb{C},$$

the Ferrand metric is the inner metric of Seittenranta's metric. Hence we have the aforementioned lower bound for the Ferrand and the K-P distances $\sigma_D(z, w)$ and $\mu_D(z, w)$:

$$s_D(z, w) \leq \sigma_D(z, w) \leq \mu_D(z, w) \quad \text{for all } z, w \in D.$$

In particular, the length of a curve in Seittenranta's metric is smaller than its length in the K-P metric.

Next we show that each circular geodesic is a geodesic line for both the Ferrand and the K-P metrics, justifying its name. Let γ be a circular geodesic in D with endpoints

a, b and $z, w \in \gamma$ be arbitrary points. Let $\gamma(z, w)$ be the subarc of γ joining z and w . We need to show that

$$\sigma_D(z, w) = \int_{\gamma(z, w)} \sigma_D(\xi) |d\xi| \quad \text{and} \quad \mu_D(z, w) = \int_{\gamma(z, w)} \mu_D(\xi) |d\xi|.$$

An easy observation shows that

$$s_D(z, w) = \log \left(1 + \frac{|a - b| |z - w|}{|a - z| |b - w|} \right) = \log \frac{|z - b| |w - a|}{|z - a| |w - b|}$$

for all $z, w \in \gamma$ such that the points a, z, w, b are in this order on γ . In particular, $s_D(z_1, z_3) = s_D(z_1, z_2) + s_D(z_2, z_3)$ for all points z_1, z_2, z_3 in this order on γ and, as a consequence, the s_D -length of $\gamma(z, w)$ is equal to $s_D(z, w)$. Then

$$\sigma_D(z, w) \leq \int_{\gamma(z, w)} \sigma_D(\xi) |d\xi| = \int_{\gamma(z, w)} \frac{|a - b|}{|a - \xi| |\xi - b|} |d\xi| = s_D(z, w) \leq \sigma_D(z, w),$$

and, similarly,

$$\mu_D(z, w) \leq \int_{\gamma(z, w)} \mu_D(\xi) |d\xi| = \int_{\gamma(z, w)} \frac{|a - b|}{|a - \xi| |\xi - b|} |d\xi| = s_D(z, w) \leq \mu_D(z, w).$$

Thus, γ is a geodesic for both the Ferrand and the K-P metrics.

Next we discuss another type of geodesic for the K-P metric. As we have mentioned above, given $z \in D$, the extremal disk B_z is also the extremal disk for all points in \hat{K}_z , where \hat{K}_z is the hyperbolic convex hull of the set $\partial B_z \cap \partial D$ in B_z . In particular, $\mu_D(\xi) = \lambda_{B_z}(\xi)$ for all $\xi \in \hat{K}_z$ and as a result, all the hyperbolic geodesics in $B_z \cap \hat{K}_z$, which are circular arcs, are also geodesics in the K-P metric μ_D . Hence through every point of D there passes a K-P geodesic which is a circular arc. Notice also that the interior of \hat{K}_z is non-empty if and only if $\text{card}(\partial B_z \cap \partial D) \geq 3$.

Now we are ready to present the result on the isometries of the K-P metric.

Theorem 4.5. *Let $f: D \rightarrow \overline{\mathbb{C}}$ be a K-P isometry. Assume that $D' = f(D)$ contains a point z' so that $\text{card}(\partial B_{z'} \cap \partial D') \geq 3$. Then f is the restriction of a Möbius transformation.*

Proof. Recall that f is conformal and f^{-1} is also a K-P isometry. Put $z = f^{-1}(z')$. Let γ be a circular arc containing z which is also a K-P geodesic segment with the property that $f(\gamma)$ is contained in the interior of $\hat{K}_{z'}$. Since all the geodesics in $\hat{K}_{z'}$ are circular arcs, so is $f(\gamma)$. Using auxiliary Möbius transformations, if necessary, we can assume that $B_z = B_{z'} = B^2(0, 1)$, that $z = z' = 0$ and that both γ and $f(\gamma)$ are subarcs of the real

interval $(-1, 1)$. Then the fact that f is an isometry implies that f is identity on γ and hence it is also identity on D , up to a Möbius map. This completes the proof. \square

There is an alternative way to prove the Theorem 4.5 based on the following result for holomorphic functions, which can be thought of as an extension of Schwarz's Lemma. This approach also extends to prove a similar theorem for the Ferrand metric (see Theorem 4.14).

Theorem 4.6 (Fact 2.1, [60]). *Let D and D' be hyperbolic regions. Assume that f is holomorphic in some neighborhood of $a \in D$ and takes values in D' . Let λ and λ' be the densities of the hyperbolic metrics in D and D' , respectively, and let $f^*[\lambda'](z) = \lambda'(f(z))|f'(z)|$ be the pullback of λ' in D to a neighborhood of a . Suppose that $f^*[\lambda'](z) \leq \lambda(z)$ for all z near a , with equality holding at $z = a$. Then $f: D \rightarrow D'$ is a holomorphic covering projection; in particular, $f^*[\lambda'] = \lambda$.*

Analytic proof of Theorem 4.5. Let $f: D \rightarrow D'$ be a K-P isometry, hence conformal. Observe first that

$$f^*[\mu_{D'}] = \mu_{D'}(f(z))|f'(z)| = \lim_{w \rightarrow z} \frac{\mu_{D'}(f(w), f(z))}{|f(w) - f(z)|} \frac{|f(w) - f(z)|}{|w - z|} = \lim_{w \rightarrow z} \frac{\mu_D(w, z)}{|w - z|} = \mu_D(z).$$

The assumption in the theorem implies that there is a point $b = f(a)$ contained in G' , where G' is the interior of \hat{K}_b and $K_b = B_b \cap \partial D'$. Then in $f^{-1}(G') \cap B_a$ we have

$$f^*[\lambda_{B_b}] = f^*[\mu_{D'}] = \mu_D \leq \lambda_{B_a},$$

with equality holding at the point $z = a$ (see the proof of [60, Theorem 4.10]). Theorem 4.6 now implies that f maps B_a conformally onto B_b and hence is a Möbius map. \square

4.3. Isometries of μ_D , σ_D and k_D ; New Results

If a disk touches the boundary of a domain in exactly k points, then we call it k -*extremal*. In this section we are interested only in domains in which every extremal disk is 2-extremal – we call such a domain also 2-extremal, as there is no danger of confusion. Examples of 2-extremal domains include parallel strips, angular sectors with angular

openings strictly less than π , annuli and many other domains and their images under Möbius mappings.

Theorem 4.7. *If D is 2-extremal domain in \mathbb{R}^2 , then circular geodesics foliate D . In particular, $\mu_D = \sigma_D$.*

Proof. We will show that each point of D lies on a circular geodesic and that circular geodesics of D are disjoint. Indeed, given an arbitrary point $x \in D$, since $\text{card}(B_x \cap \partial D) = 2$, the interior of the set \hat{K}_x is empty, whence \hat{K}_x is a circular geodesic containing x . Next if γ_1 and γ_2 are two circular geodesics in D and if $x \in \gamma_1 \cap \gamma_2$, then the endpoints of γ_1 and γ_2 belong to the set $\partial B_x \cap \partial D$. Since $\text{card}(B_x \cap \partial D) = 2$, we conclude that $\gamma_1 = \gamma_2$. The second part of the theorem now follows from the fact that $\mu_D(x) = \sigma_D(x)$ whenever x lies on a circular geodesic (see Section 4.2.2). \square

Herron, Ibragimov and Minda proved that every planar 2-extremal domain is either simply or doubly connected, see [60]. Given a 2-extremal disk B in a domain D , we denote the unique circular geodesic in B by $\gamma(B)$. The (Euclidean) midpoint of the circular geodesic is called the hyperbolic center of B and denoted by $\text{HC}(B)$. Let $E(D)$ be the set of all 2-extremal disks in D .

Definition 4.8. The set of hyperbolic centers of disks in $E(D)$ is called the *hyperbolic medial axis* of the domain D and is denoted by $\text{HMA}(D)$.

The hyperbolic medial axis is a modification of the usual medial axis, whose definition was presented in Section 4.2. In certain respects the hyperbolic medial axis is better behaved than the medial axis; for example, this is the case for smoothness and localization properties. A more thorough investigation of these issues is underway [58].

Theorem 4.9. *If $D \subset \mathbb{R}^2$ is a 2-extremal domain, then $\text{HMA}(D)$ is locally the graph of a C^1 curve. If B is a 2-extremal disk in D , then the circular geodesic $\gamma(B)$ and $\text{HMA}(D)$ are orthogonal at the hyperbolic center $\text{HC}(B)$.*

Proof. Let B be a 2-extremal disk corresponding to the boundary point a and b . Let B_a be the disk in B with a and $\text{HC}(B)$ as boundary points; B_b is defined similarly. Note that

B_a and B_b are horodisks in B . The circle ∂B_a is tangent to ∂B at a , so it is orthogonal to $\gamma(B)$ there, hence also at $\text{HC}(B)$. Thus both ∂B_a and ∂B_b are orthogonal to $\gamma(B)$ at $\text{HC}(B)$ and, in particular, $\text{HMA}(D) \cap U \subset U \setminus (B_a \cup B_b)$ for some sufficiently small neighborhood U of $\text{HC}(B)$. It is clear that $\text{HMA}(D)$ is orthogonal to $\gamma(B)$ and has smoothness C^1 at $\text{HC}(B)$. \square

For metric densities which are at least C^2 smooth it is well-known that geodesics are locally unique (i.e., through a given point in a given direction there is only one geodesic). For metrics defined by densities with less smoothness this is not the case. For instance for the quasihyperbolic metric in the strip $\{x \in \mathbb{R}^2: |x_2| < 1\}$ we know that a geodesic consists of a circular arc, a segment lying in the real axis and a second circular arc (any two of these three pieces may of course be degenerate). In particular, geodesics are not locally unique on the real axis in the real direction.

It was shown in Theorem 4.7 that there is a unique circular geodesic through every point in a 2-extremal domain. We next prove a stronger statement: there is no geodesic which is tangent to a circular geodesic.

Lemma 4.10. *Smooth geodesics of the K–P metric are locally unique in 2-extremal domains in the direction of the circular geodesic.*

Proof. Using an auxiliary Möbius mapping we may restrict ourselves to the circular geodesic $(-1, 1) \subset \mathbb{R}$. More specifically, we show that there is no other geodesic through the origin which is parallel to the real axis there.

As before we denote by μ_D the density of the K–P metric in our domain. By $\tilde{\mu}$ we denote the density of the K–P metric in the domain $\{x \in \mathbb{R}^2: |x_1| < 1\}$. Obviously, $\tilde{\mu}(x) = 2(1 - x_1^2)^{-1}$. As in the proof of Theorem 4.9 we find that the level sets of μ_D are constrained by a pair of balls. We restrict our attention to a small neighborhood of the origin. Then the radii of these balls are greater than some constant $r > 0$, so we see that the level-sets of μ_D are approximated by the level-sets of $\tilde{\mu}$ near the real axis. More

precisely,

$$\begin{aligned}\frac{1}{\mu_D(x)} &\geq \frac{1}{\tilde{\mu}(|x_1| + \Delta x)} = 1 - \left(|x_1| + r\left(1 - \sqrt{1 - x_2^2}\right)\right)^2 \\ &\geq 1 - (|x_1| + rx_2^2)^2 = 1 - x_1^2 + O(|x_1| x_2^2 + x_2^4).\end{aligned}$$

Here Δx is the maximal distance between the level-set of μ_D and $\tilde{\mu}$ at distance x_2 from the real axis. A similar lower bound applies, so we have

$$|\mu_D(x) - (1 - x_1^2)^{-1}| \leq C|x_1| x_2^2$$

provided $x_2 = O(x_1)$.

Now suppose that there is a second smooth geodesic through the origin that is parallel to the real axis. Locally such a geodesic can be represented by

$$y = f(x) = c_2 x^2 + O(x^3).$$

We also define $F: \mathbb{R} \rightarrow \mathbb{R}^2$ by $F(x) = (x, f(x))$. We assume that $c_2 > 0$; the cases of negative coefficient or lower order leading term are similar. We will show that for small enough $\epsilon > 0$, the segment $L_1 = [0, F(\epsilon)]$ is shorter than the curve $L_2 = \{F(x): 0 < x < \epsilon\}$. Thus the latter curve is certainly not a geodesic, which proves local uniqueness. We also introduce the function $G: \mathbb{R} \rightarrow \mathbb{R}^2$ which parameterizes L_1 : $G(x) = (x, \frac{x}{\epsilon} f(\epsilon))$.

We start by calculating the length of L_2 :

$$\mu_D(L_2) = \int_{L_2} \mu_D(z) dz = \int_0^\epsilon \mu(F(x)) \sqrt{1 + f'(x)^2} dx.$$

We know that

$$\mu(F(x)) = 1 - x^2 + O(x(c_2 x^2)^2) = 1 - x^2 + O(x^5).$$

Thus we find that

$$\begin{aligned}\mu(L_2) &= \int_0^\epsilon \mu(F(x)) (1 + \tfrac{1}{2} f'(x)^2 + O(f'(x)^4)) dx \\ &= \int_0^\epsilon (1 - x^2 + O(x^5)) (1 + 2c_2^2 x^2 + O(x^4)) dx \\ &= \int_0^\epsilon (1 + (2c_2^2 - 1)x^2 + O(x^4)) dx \\ &= \epsilon + (\tfrac{2}{3}c_2^2 - \tfrac{1}{3})\epsilon^3 + O(\epsilon^5).\end{aligned}$$

For L_1 we calculate

$$\begin{aligned}
\mu(L_1) &= \int_0^\epsilon \mu(G(x)) \sqrt{1 + (f(\epsilon)/\epsilon)^2} dx \\
&= (1 + \tfrac{1}{2}c_2^2\epsilon^2 + O(\epsilon^4)) \int_0^\epsilon (1 - x^2 + O(x(\tfrac{x}{\epsilon}f(\epsilon))^2)) dx \\
&= (1 + \tfrac{1}{2}c_2^2\epsilon^2)(\epsilon - \tfrac{1}{3}\epsilon^3) + O(\epsilon^4) \\
&= \epsilon + (\tfrac{1}{2}c_2^2 - \tfrac{1}{3})\epsilon^3 + O(\epsilon^4).
\end{aligned}$$

A comparison with the expression for $\mu(L_2)$ shows that $\mu(L_1) < \mu(L_2)$ whenever ϵ is sufficiently small, so L_2 is not a geodesic. \square

Lemma 4.11. *Let D be 2-extremal and doubly connected domain in \mathbb{R}^2 . Then a simple C^1 curve γ in D which is not contractible is orthogonal to a circular geodesic at some point.*

Proof. The claim clearly holds in the special case $D^c \subset \mathbb{R}$. Thus we assume that D has a non-degenerate boundary component. Since our domain is doubly connected, it is a ring domain of the type $G \setminus K$, where G is open and K is a closed subset of G . We assume without loss of generality that $\infty \notin \overline{G}$.

By Theorem 4.7, D is foliated by circular geodesics. We think of circular geodesics as directed curves which start at K . We denote the circular geodesic through x by C_x , and the tangent of this curve at x by T_x . If γ is not orthogonal to any of the circular geodesics, then either $T_x \cdot \nabla \gamma(x) > 0$ for all x , or $T_x \cdot \nabla \gamma(x) < 0$ for all x , where ∇ denotes the gradient vector.

A point $x \in \gamma$ divides C_x in two parts whose lengths are denoted by $l_K(x)$ and $l_G(x)$. Let L_r be the set of points $x \in D$ such that $l_G(x) = r l_K(x)$. As in Theorem 4.9, we find that the simple closed curve L_r is C^1 and orthogonal to all circular geodesics. Let $x_0 \in \gamma$ and $r = l_G(x_0)/l_K(x_0)$. Then $x_0 \in L_r$. Now if $T_x \cdot \nabla \gamma(x) > 0$ for all x , then we see that γ will not cross L_r again. Therefore γ cannot be a closed curve, which is a contradiction. The same conclusion holds if $T_x \cdot \nabla \gamma(x) < 0$ for all x . Thus there must be a point of orthogonality between the curves. \square

We are now ready to prove the main result of this chapter. Note that this result combined with the results from Section 4.2.2 takes care of the isometry problem for the K-P metric, except in some cases of simply connected planar domains.

Theorem 4.12. *Let D be 2-extremal and doubly connected domain in \mathbb{R}^2 . Then every isometry of μ_D is a Möbius mapping.*

Proof. Let f an isometry of μ_D . As in the previous proof, we may assume that $D = G \setminus K$, where G is open and bounded, and K is a closed subset of G . So every circular geodesic connects ∂G to ∂K . By Theorem 4.9, for each 2-extremal disk B the hyperbolic medial axis $\text{HMA}(D)$ is orthogonal to the circular geodesic $\gamma(B)$ at the hyperbolic center $\text{HC}(B)$. We recall that every isometry is a conformal mapping in the usual, Euclidean sense. Hence, if we show that the isometry coincides with a Möbius map on an arc of a circle, then it follows that f is Möbius.

By [60, Theorem B] we know that every isometry of a domain which is not 2-extremal is Möbius. Note that f^{-1} is also an isometry. Thus, it is enough for us to consider the case when $f(D)$ is also 2-extremal. Now, Theorem 4.9 shows that $\text{HMA}(D)$ is a C^1 curve and thus image of $\text{HMA}(D)$ under f is a simple closed C^1 curve in $f(D)$. By Lemma 4.11, $f(\text{HMA}(D))$ is orthogonal to a circular geodesic C' at some point, say $f(x)$. Since f^{-1} is an isometry, we find that $f^{-1}(C')$ is a geodesic line in D . Since f^{-1} is conformal, we see that $f^{-1}(C')$ is orthogonal to $\text{HMA}(D)$ at x . But by Theorem 4.9, $\text{HMA}(D)$ is orthogonal to a circular geodesic C at x , and since geodesics are unique by Lemma 4.10, it follows that $C = f^{-1}(C')$. Since f maps an arc of a circle to an arc of a circle and is a K-P isometry, we easily conclude as in the first proof of Theorem 4.5 that f coincides with a Möbius map on C , which completes the proof. \square

We end this section by presenting two results on the isometries of the Ferrand and the quasihyperbolic metrics.

Corollary 4.13. *Let $K \subset \mathbb{R}^2$ be convex, closed and non-degenerate, and set $D = \mathbb{R}^2 \setminus K$. Then every isometry $f: D \rightarrow \mathbb{R}^2$ of the quasihyperbolic metric is a similarity mapping.*

Proof. It is clear that $\text{MA}(D) = \emptyset$. From Proposition 4.3 we see that this implies $\text{MA}(f(D)) = \emptyset$. From this it follows easily that $f(D) = \mathbb{R}^2 \setminus K'$, where K' is convex. Moreover, we easily see that $k_G = \mu_G$ if G is the complement of a convex closed set. Thus $k_D = \mu_D$ and $k_{f(D)} = \mu_{f(D)}$. Therefore f is an isometry of the K-P metric, so the claim follows from the previous theorem. \square

The next result deals with Ferrand isometries in the special case of a domain with a circular arc as part of its boundary. Although this is quite a restrictive assumption, we would like to point out that so far no results whatsoever have been derived for the isometries of this metric.

Theorem 4.14. *Let $D \subset \mathbb{R}^2$ be a domain, and $f: D \rightarrow \mathbb{R}^2$ be a Ferrand isometry. Assume that there exists a disk $B \subset D$ with the property that $\partial B \cap \partial D$ contains an arc γ . Then f is the restriction of a Möbius transformation.*

We will prove this claim using Theorem 4.6. In order to conform with the notation of that theorem, we will actually prove the following claim, which is easily seen to be equivalent to the previous theorem.

Lemma 4.15. *Let $D \subset \mathbb{R}^2$ be a domain, and $f: D \rightarrow \mathbb{R}^2$ be a Ferrand isometry. Assume that there exists a disk $B' \subset D'$, $D' = f(D)$, with the property that $\partial B' \cap \partial D'$ contains an arc γ' . Then f is the restriction of a Möbius transformation.*

Proof. Since f is conformal we see as in the second proof of Theorem 4.5 that

$$f^*[\sigma_{D'}] = \sigma_{D'}(f(z))|f'(z)| = \sigma_D(z).$$

Let G' be the interior of the hyperbolic convex hull of γ' in B' . Then one can easily see that $\sigma_{D'}(x) = \lambda_{B'}(x)$ for all $x \in G'$. Let $G = f^{-1}(G')$. First we claim that there exists a point $a \in G$ with $\sigma_D(a) = \mu_D(a)$. Using this claim we obtain

$$f^*[\lambda_{B'}] = f^*[\sigma_{D'}] = \sigma_D \leq \mu_D \leq \lambda_{B_a},$$

with equality holding at the point $x = a$. The proof is then completed by invoking Theorem 4.6. Thus, it remains to prove the claim.

Observe first that if there exist points $x, y \in G$ with $\hat{K}_x \cap \hat{K}_y = \emptyset$, then due to the connectedness of G there exists a point $a \in G \cap \partial \hat{K}_x$ (i.e., a lies on a circular geodesic and hence $\sigma_D(a) = \mu_D(a)$; see Section 4.2.2). We can now assume that $\hat{K}_x = \hat{K}_y$ for all $x, y \in G$. In particular, all the points of G share a common extremal disk, which we can assume to be the unit disk B about the origin. Put $K = \partial B \cap \partial D$, and let \hat{K} be the hyperbolic convex hull of K in B . Note that $\hat{K}_x = \hat{K}$ for each $x \in G$. Since $G \subset \hat{K} \subset B$, B is not 2-extremal. Hence we have a conformal map f^{-1} of B' into B with a property that $|f^{-1}(z)| \rightarrow 1$ as $z \rightarrow \gamma'$. Then the Schwarz Reflection Principle implies that f^{-1} has an analytic continuation onto γ' and by the identity theorem it can not map γ' onto a single point. Thus, the set $f^{-1}(\gamma') \subset \partial B \cap \partial D$ contains an open arc, say γ . Then $\sigma_D(a) = \mu_D(a)$ for each point of the hyperbolic convex hull of γ , as required. \square

CHAPTER 5

CERTAIN CLASSES OF UNIVALENT FUNCTIONS AND RADIUS PROBLEMS

This chapter is devoted to the study of certain subclasses of the class \mathcal{S} of univalent analytic functions with an aim to obtain coefficient conditions for functions to be in some subclasses of \mathcal{S} and radius problems. Section 5.1 is introductory in nature. Section 5.2 contains some lemmas those are require to prove our results. In Section 5.3, we obtain some coefficient conditions for functions in $\mathcal{S}_p(\alpha)$ in series form. Section 5.4 discusses radius problems. In Section 5.5, we obtain some conditions for functions to be in the class $\mathcal{U}(\lambda, \mu)$. Section 5.6 concludes with some observations.

The results of this chapter are from the published paper: **S. Ponnusamy and S.K. Sahoo** (2006) Study of some subclasses of univalent functions and their radius properties. *Kodai Math. J.* **29**(3), 391–405.

5.1. Introduction and Preliminaries

Recall that a function $f \in \mathcal{A}$ is said to be in $\mathcal{U}(\lambda, \mu)$ if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu+1} - 1 \right| \leq \lambda \quad (|z| < 1)$$

for some $\lambda \geq 0$ and $\mu > -1$. We set $\mathcal{U}(\lambda, 1) = \mathcal{U}(\lambda)$, and $\mathcal{U}(1) = \mathcal{U}$. In [88], the authors studied a subclass $\mathcal{P}(2\lambda)$ of $\mathcal{U}(\lambda)$, consisting of functions f for which

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2\lambda \quad (|z| < 1).$$

We have the strict inclusion $\mathcal{P}(2) \subsetneq \mathcal{U} \subsetneq \mathcal{S}$, see [1, 85, 94]. Moreover, a close connection between the classes $\mathcal{P}(2\lambda)$ and $\mathcal{U}(\lambda)$ is given by $\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda)$, see [87, 88].

At this place it is important to remark that functions in \mathcal{U} need not be starlike (see [89]). Also functions in \mathcal{S}^* need not be in \mathcal{U} (see [31]). Extremal functions of many subclasses of \mathcal{S} are in \mathcal{U} (see [89]). For instance if

$$L = \left\{ z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\},$$

then each function in this collection is in $\mathcal{U} \cap \mathcal{S}^*$. In [86, 100, 101], the authors considered the problem of finding conditions on λ and μ so that each function in $\mathcal{U}(\lambda, \mu)$ is starlike or in some subsets of \mathcal{S} . For example, Ponnusamy and Singh [100] have shown that

$$\mathcal{U}(\lambda, \mu) \subseteq \mathcal{S}^* \quad \text{if } \mu < 0 \text{ and } 0 \leq \lambda \leq \frac{1 - \mu}{\sqrt{(1 - \mu)^2 + \mu^2}} := \lambda^*(\mu)$$

and in [86], Obradović proved that the above inclusion continues to hold for $0 < \mu \leq 1$ and with the same bound for λ . The sharpness part of these results may be obtained as a consequence of results from [110]. However, it is not known whether each function f in $\mathcal{U}(1, \mu)$ (or more generally, $\mathcal{U}(\lambda, \mu)$ with $\lambda^*(\mu) < \lambda \leq 1$) is univalent in \mathbb{D} for certain values of μ in the open interval $(0, 1)$. On the other hand, according to a result due to Aksentiev [1] (see also Ozaki and Nunokawa [94] for a reformulated version as given by \mathcal{U}), we have the inclusion $\mathcal{U}(\lambda) \subset \mathcal{S}$ for $0 \leq \lambda \leq 1$. We see that the Koebe function $z/(1 - z)^2$ belongs to \mathcal{U} but does not belong to $\mathcal{S}^*(\alpha)$ for any $\alpha > 0$. In fact, the bounded function $z + z^2/2$ belongs to \mathcal{U} but not in $\mathcal{S}^*(\alpha)$ for any $\alpha > 0$. That is, $\mathcal{U} \not\subseteq \mathcal{S}^*(\alpha)$ for any $\alpha > 0$. Thus, $\mathcal{U} \subsetneq \mathcal{S}$ and the inclusion is strict as functions in \mathcal{S} are not necessarily in \mathcal{U} . Further work on these classes, including some interesting generalizations of these classes, may be found in [87, 91, 103].

A function $f \in \mathcal{S}^*(\alpha)$ is said to be in $\mathcal{T}^*(\alpha)$ if it can be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k.$$

Functions of this form are discussed in detail by Silverman [115] and others [116].

In this chapter we shall be mainly concerned with functions $f \in \mathcal{A}$ of the form

$$(5.1) \quad \left(\frac{z}{f(z)} \right)^\mu = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D},$$

where $(z/f(z))^\mu$ represents principal powers (i.e. the principal branch of $(z/f(z))^\mu$ is chosen). The class of functions f of this form for which $b_n \geq 0$ is especially interesting

and deserves separate attention. We remark that if $f \in \mathcal{S}$ then $z/f(z)$ is nonvanishing and hence, $f \in \mathcal{S}$ may be expressed as

$$f(z) = \frac{z}{g(z)}, \quad \text{where } g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}.$$

These two representations are convenient for our investigation. Finally, we introduce a subclass $\mathcal{S}_p(\alpha)$, $-1 \leq \alpha \leq 1$, of starlike functions in the following way [108]:

$$\mathcal{S}_p(\alpha) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)} - \alpha, \quad z \in \mathbb{D} \right\}.$$

Geometrically, $f \in \mathcal{S}_p(\alpha)$ if and only if the domain values of $zf'(z)/f(z)$, $z \in \mathbb{D}$, is the parabolic region

$$(\operatorname{Im} w)^2 \leq (1 - \alpha)[2\operatorname{Re} w - (1 + \alpha)].$$

In [108], Rønning has shown that the class $\mathcal{S}_p(\alpha)$ must contain non-univalent functions if $\alpha < -1$, and $\mathcal{S}_p(\alpha) \subset \mathcal{S}^*$ if $-1 \leq \alpha \leq 1$. We set $\mathcal{S}_p(0) = \mathcal{S}_p$. The class of uniformly convex functions was introduced by Goodman in [45] (see also [46] where Goodman studied the class of uniformly starlike functions). Later Rønning [109] studied these classes along with the class \mathcal{S}_p . Moreover, from the work of Rønning [109], it follows easily that $f(z) = z + a_n z^n$ is in $\mathcal{S}_p(\alpha)$ if and only if $(2n - 1 - \alpha)|a_n| \leq 1 - \alpha$.

We refer to Section 1.4 for the definition of radius problem. There are many results of this type in the theory of univalent functions. For example, the \mathcal{S}_p radius in \mathcal{S}^* was found by Rønning in [109] to be $1/3$. Also, $\mathcal{P}(2)$ radius in \mathcal{U} has been obtained by Obradović and Ponnusamy in [90] and is given by $2/3$. At this place, it is appropriate to recall the following result:

Theorem A. [109, Theorem 4] *If $f \in \mathcal{S}$, then $\frac{1}{r}f(rz) \in \mathcal{S}_p$ if and only if $0 < r \leq 0.33217\dots$*

5.2. Lemmas

For the proof of our results, we need the following result (see [44, Theorem 11 in p.193 of Vol-2]) which reveals the importance of the area theorem in the theory of univalent functions.

Lemma 5.1. *Let $\mu > 0$ and $f \in \mathcal{S}$ be of the form (5.1). Then we have*

$$\sum_{n=1}^{\infty} (n - \mu) |b_n|^2 \leq \mu.$$

Next we recall the well-known coefficient condition that is sufficient for functions to be in $\mathcal{U}(\lambda)$ or $\mathcal{P}(2\lambda)$ or $\mathcal{S}^*(\alpha)$, respectively.

Lemma 5.2. [91] *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in \mathbb{D} and $f(z) = z/\phi(z)$. If $\sum_{n=2}^{\infty} (n-1) |b_n| \leq \lambda$, then we have*

(a) $f \in \mathcal{U}(\lambda)$

(b) $f \in \mathcal{U}(\lambda) \cap \mathcal{S}^*$ for $0 < \lambda \leq \frac{\sqrt{2-|b_1|^2}-|b_1|}{2} = \lambda_*(f)$;

(c) Further, if $\sum_{n=2}^{\infty} n(n-1) |b_n| \leq 2\lambda$, then we have $f \in \mathcal{P}(2\lambda)$.

In [106], it was shown that if $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is a non-vanishing analytic function in \mathbb{D} and $f(z) = z/\phi(z)$, then $f \in \mathcal{S}^*(\alpha)$, $0 \leq \alpha \leq 1$, whenever

$$\sum_{k=2}^{\infty} (k-1+\alpha) |b_k| \leq \begin{cases} 1-\alpha-(1-\alpha)|b_1| & \text{if } 0 \leq \alpha \leq 1/2 \\ 1-\alpha-\alpha|b_1| & \text{if } 1/2 \leq \alpha \leq 1. \end{cases}$$

5.3. Coefficient Conditions for Functions in $\mathcal{S}_p(\alpha)$

Theorem 5.3. *If a function f of the form (5.1) with $b_n \geq 0$ and $\mu > 0$ is in $\mathcal{S}_p(\alpha)$, we then have*

$$(5.2) \quad \sum_{n=1}^{\infty} (2n - \mu(1-\alpha)) b_n \leq \mu(1-\alpha).$$

Proof. Let $f \in \mathcal{S}_p(\alpha)$. Now, it is easy to see that

$$(5.3) \quad z \frac{d}{dz} \left(\frac{z}{f(z)} \right)^{\mu} = \mu \left[\left(\frac{z}{f(z)} \right)^{\mu} - \left(\frac{z}{f(z)} \right)^{\mu+1} f'(z) \right].$$

Using the identity (5.3), we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) - \alpha \Leftrightarrow \left| \frac{-\frac{z}{\mu} \frac{d}{dz} \left(\frac{z}{f(z)} \right)^{\mu}}{\left(\frac{z}{f(z)} \right)^{\mu}} \right| \leq \operatorname{Re} \frac{\left(\frac{z}{f(z)} \right)^{\mu} - \frac{z}{\mu} \frac{d}{dz} \left(\frac{z}{f(z)} \right)^{\mu}}{\left(\frac{z}{f(z)} \right)^{\mu}} - \alpha.$$

Since f is in the form (5.1), the last inequality may be equivalently written as

$$\frac{1}{\mu} \left| \frac{-\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n} \right| \leq \operatorname{Re} \left(1 - \frac{1}{\mu} \frac{\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n} \right) - \alpha.$$

If $z \in \mathbb{D}$ is real and tends to 1^- through reals, then from the last inequality we have

$$\frac{1}{\mu} \left(\frac{\sum_{n=1}^{\infty} n b_n}{1 + \sum_{n=1}^{\infty} b_n} \right) \leq 1 - \alpha - \frac{1}{\mu} \left(\frac{\sum_{n=1}^{\infty} n b_n}{1 + \sum_{n=1}^{\infty} b_n} \right),$$

from which we obtain the desired inequality (5.2). \square

The case $\mu = 1$ leads to

Corollary 5.4. *Let $f \in \mathcal{S}_p(\alpha)$ be such that $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ with $b_n \geq 0$. Then we have*

$$\sum_{n=1}^{\infty} (2n - 1 + \alpha) b_n \leq 1 - \alpha.$$

Theorem 5.5. *Let $z/f(z)$ be a nonvanishing analytic function of the form (5.1) with $\mu > 0$. Then the condition*

$$(5.4) \quad \sum_{n=1}^{\infty} (2n + \mu(1 - \alpha)) |b_n| \leq \mu(1 - \alpha)$$

is sufficient for f to be in the class $\mathcal{S}_p(\alpha)$.

Proof. As in the proof of Theorem 5.3, we notice that

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) - \alpha$$

is equivalent to

$$\left| -\frac{\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n} \right| \leq \mu(1 - \alpha) - \operatorname{Re} \left(\frac{\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n} \right).$$

Thus, to show that f is in $\mathcal{S}_p(\alpha)$, it suffices to show that the quotient

$$-\frac{\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n}$$

lies in the parabolic region

$$(\operatorname{Im} w)^2 \leq \mu(1 - \alpha)[\mu(1 - \alpha) + 2\operatorname{Re} w].$$

Geometrically, this condition holds if we can show that

$$(5.5) \quad \left| \frac{\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n} \right| \leq \frac{\mu(1 - \alpha)}{2}, \quad z \in \mathbb{D}.$$

From the condition (5.4), we obtain that

$$\sum_{n=1}^{\infty} (2n + \mu(1 - \alpha)) |b_n| |z|^n \leq \mu(1 - \alpha)$$

and so

$$\sum_{n=1}^{\infty} n |b_n| |z|^n \leq \frac{\mu(1 - \alpha)}{2} \left(1 - \sum_{n=1}^{\infty} |b_n| |z|^n \right).$$

In view of this inequality, we deduce that

$$\left| \frac{\sum_{n=1}^{\infty} n b_n z^n}{1 + \sum_{n=1}^{\infty} b_n z^n} \right| \leq \frac{\mu(1 - \alpha)}{2} \left(\frac{1 - \sum_{n=1}^{\infty} |b_n| |z|^n}{1 - \sum_{n=1}^{\infty} |b_n| |z|^n} \right) = \frac{\mu(1 - \alpha)}{2}$$

which is exactly the inequality (5.5) and therefore, $f \in \mathcal{S}_p(\alpha)$. \square

Corollary 5.6. *Let $z/f(z)$ be a nonvanishing analytic function in \mathbb{D} of the form $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$. Then the condition*

$$\sum_{n=1}^{\infty} (2n + 1 - \alpha) |b_n| \leq 1 - \alpha$$

is sufficient for f to be in the class $\mathcal{S}_p(\alpha)$.

The case $\alpha = 0$ of Corollaries 5.4 and 5.6 has been obtained recently by Obradović and Ponnusamy [90].

5.4. Radius Problems

Theorem 5.7. *If $f \in \mathcal{S}$ is given by (5.1) with $0 < \mu < 1$, then $\frac{1}{r}f(rz) \in \mathcal{S}_p(\alpha)$ for $0 < r \leq r_0$, where r_0 is the root of the integral equation*

$$(5.6) \quad \frac{4r^2(1 + \mu(2 - \alpha)(1 - r^2))}{(1 - r^2)^2} + \frac{r^2\mu^2(3 - \alpha)^2}{1 - \mu} \int_0^1 \frac{dt}{1 - r^2 t^{1/(1-\mu)}} = \mu(1 - \alpha)^2.$$

Proof. Let $f \in \mathcal{S}$ be given by (5.1) with $0 < \mu < 1$. Then $z/f(z)$ is nonvanishing in \mathbb{D} and for $0 < r \leq 1$, we have

$$\left(\frac{z}{\frac{1}{r}f(rz)} \right)^{\mu} = 1 + (b_1 r)z + (b_2 r^2)z^2 + \dots.$$

If

$$(5.7) \quad S := \sum_{n=1}^{\infty} (2n + \mu(1 - \alpha)) |b_n| r^n \leq \mu(1 - \alpha)$$

for some r , then $\frac{1}{r}f(rz) \in \mathcal{S}_p(\alpha)$, by Theorem 5.5. Now, using the Cauchy-Schwarz inequality and Lemma 5.1, we see that

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \sqrt{n-\mu} |b_n| \frac{2n + \mu(1-\alpha)}{\sqrt{n-\mu}} r^n \\
&\leq \left(\sum_{n=1}^{\infty} (n-\mu) |b_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \frac{(2n + \mu(1-\alpha))^2}{n-\mu} r^{2n} \right)^{\frac{1}{2}} \\
&\leq \sqrt{\mu} \left(\sum_{n=1}^{\infty} \frac{(2n + \mu(1-\alpha))^2}{n-\mu} r^{2n} \right)^{\frac{1}{2}} \\
&= \sqrt{\mu} \left(\sum_{n=1}^{\infty} \frac{(2n + \mu(1-\alpha))^2 - \mu^2(3-\alpha)^2}{n-\mu} r^{2n} + \mu^2(3-\alpha)^2 \sum_{n=1}^{\infty} \frac{r^{2n}}{n-\mu} \right)^{\frac{1}{2}} \\
&= \sqrt{\mu} \left(\sum_{n=1}^{\infty} 4(n + \mu(2-\alpha)) r^{2n} + \mu^2(3-\alpha)^2 \sum_{n=1}^{\infty} \frac{r^{2n}}{n-\mu} \right)^{\frac{1}{2}} \\
&= \sqrt{\mu} \left(\frac{4r^2(1 + \mu(2-\alpha)(1-r^2))}{(1-r^2)^2} + \frac{r^2\mu^2(3-\alpha)^2}{1-\mu} \int_0^1 \frac{dt}{1-r^2t^{1/(1-\mu)}} \right)^{\frac{1}{2}}.
\end{aligned}$$

In particular, if the last expression is less than or equal to $\mu(1-\alpha)$, then (5.7) holds which gives the condition (5.6). \square

In the case $\mu = 1$, Theorem 5.7 takes the following form which needs a special attention as we see that the radius quantity depends on the second coefficient of the given function f .

Theorem 5.8. *If $f \in \mathcal{S}$ is of the form $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, then $\frac{1}{r}f(rz) \in \mathcal{S}_p(\alpha)$ for $0 < r \leq r_0$, where r_0 , which depends on the second coefficient of f , is the root of the equation*

$$\frac{4r^4(1 + (3-\alpha)(1-r^2))}{(1-r^2)^2} - (3-\alpha)^2 r^2 \ln(1-r^2) = (1-\alpha - (3-\alpha)(r/2)|f''(0)|)^2.$$

Proof. Note that, for $f \in \mathcal{S}$ satisfying $z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$, we have $b_1 = -f''(0)/2$. Proceeding exactly as in the proof of Theorem 5.7 (but with $\mu = 1$) and by considering summation to run from 2 to ∞ , we obtain the required conclusion. So we omit the details. \square

We remark that, the case $\alpha = 0$ of Theorem 5.8 is due to Obradović and Ponnusamy [90].

Now we prove a generalized version of Lemma 5.2(a) which is useful to prove our next result.

Lemma 5.9. *Let $0 \leq \alpha < 1$ and $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in \mathbb{D} satisfying the coefficient condition*

$$(5.8) \quad \sum_{n=1}^{\infty} (n-1+\alpha) |b_n| \leq \lambda(1-\alpha).$$

Then the function f defined by the equation $(z/f(z))^{1-\alpha} = \phi(z)$ is in $\mathcal{U}(\lambda, 1-\alpha)$.

Proof. Let f be given by $(z/f(z))^{1-\alpha} = \phi(z)$, where $\phi(z) \neq 0$ in \mathbb{D} , and we choose here the principal branch so that $(z/f(z))^{1-\alpha}$ at $z = 0$ is 1. Then the power series representation of ϕ and the coefficient condition (5.8), lead to

$$\left| \left(\frac{z}{f(z)} \right)^{2-\alpha} f'(z) - 1 \right| = \left| -\frac{1}{1-\alpha} \sum_{n=1}^{\infty} (n-1+\alpha) b_n z^n \right| \leq \lambda$$

and therefore, by the definition of the class, f is in $\mathcal{U}(\lambda, 1-\alpha)$. □

The following result determines the $\mathcal{U}(\lambda, \mu)$ radius in \mathcal{S} .

Theorem 5.10. *Suppose that $f \in \mathcal{S}$, $0 \leq \alpha < 1$, $\lambda > 0$ and*

$$r_{\alpha, \lambda} = \frac{\lambda \sqrt{2(1-\alpha)}}{\left[\sqrt{(\alpha + 2\lambda^2(1-\alpha))^2 + 4\lambda^2(1-\alpha)^2(1-\lambda^2)} + (\alpha + 2\lambda^2(1-\alpha)) \right]^{1/2}}.$$

Then we have $\frac{1}{r} f(rz) \in \mathcal{U}(\lambda, 1-\alpha)$ for

$$(5.9) \quad 0 < r \leq r_{\alpha, \lambda}.$$

In particular, $\frac{1}{r} f(rz) \in \mathcal{U}(1, 1-\alpha)$ for $0 < r \leq \sqrt{(1-\alpha)/(2-\alpha)}$.

Proof. Let $f \in \mathcal{S}$. Then $z/f(z) \neq 0$ in \mathbb{D} . So, we may assume f is of the form

$$(5.10) \quad \left(\frac{z}{f(z)} \right)^{1-\alpha} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

Now, Lemma 5.1 gives

$$\sum_{n=1}^{\infty} (n-1+\alpha) |b_n|^2 \leq 1-\alpha.$$

On the other side, for $0 < r \leq 1$, we obtain from (5.10) that

$$\left(\frac{z}{\frac{1}{r}f(rz)} \right)^{1-\alpha} = 1 + \sum_{n=1}^{\infty} (b_n r^n) z^n.$$

According to Lemma 5.9, it suffices to verify the inequality

$$\sum_{n=1}^{\infty} (n-1+\alpha) |b_n r^n| \leq \lambda(1-\alpha)$$

for $0 < r \leq r_{\alpha, \lambda}$. Now, as before, we have

$$\begin{aligned} \sum_{n=1}^{\infty} (n-1+\alpha) |b_n r^n| &\leq \left(\sum_{n=1}^{\infty} (n-1+\alpha) |b_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} (n-1+\alpha) r^{2n} \right)^{1/2} \\ &\leq \sqrt{1-\alpha} \left(\frac{r^4}{(1-r^2)^2} + \alpha \frac{r^2}{1-r^2} \right)^{1/2} \\ &= \sqrt{1-\alpha} \left(\frac{r}{1-r^2} \right) (\alpha + (1-\alpha)r^2)^{1/2} \\ &\leq \lambda(1-\alpha), \end{aligned}$$

if $\frac{r}{1-r^2} \sqrt{\alpha + (1-\alpha)r^2} \leq \lambda \sqrt{1-\alpha}$. Note that

$$\frac{r}{1-r^2} \sqrt{\alpha + (1-\alpha)r^2} \leq \lambda \sqrt{1-\alpha}$$

is equivalent to (5.9), and so we complete the proof. \square

5.5. Conditions for Functions to be in $\mathcal{U}(\lambda, \mu)$

To present our next result, we consider the class of functions of Bazilevič type, see [76, 114]. The result is simple and surprising as it identifies a subclass which lies in $\mathcal{U}(\lambda, \mu)$. This generalizes the result of Obradović and Ponnusamy, see [90, Theorem 5].

Theorem 5.11. *Let $0 < \mu \leq 1$. If $f \in \mathcal{S}$ is given by (5.1) with $b_n \geq 0$, and satisfies the condition that $\operatorname{Re} \left(f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) > 0$. Then $f \in \mathcal{U}(1, \mu)$.*

Proof. Using the equation (5.3), we notice that

$$\begin{aligned}
\operatorname{Re} \left(f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) > 0 &\Leftrightarrow \operatorname{Re} \frac{\frac{zf'(z)}{f(z)}}{\left(\frac{z}{f(z)} \right)^{\mu}} > 0 \\
&\Leftrightarrow \operatorname{Re} \frac{\left(\frac{z}{f(z)} \right)^{\mu} - \frac{z}{\mu} \frac{d}{dz} \left(\frac{z}{f(z)} \right)^{\mu}}{\left(\frac{z}{f(z)} \right)^{2\mu}} > 0 \\
&\Leftrightarrow \operatorname{Re} \frac{1 + \sum_{n=1}^{\infty} (1 - n/\mu) b_n z^n}{(1 + \sum_{n=1}^{\infty} b_n z^n)^2} > 0.
\end{aligned}$$

Since $b_n \geq 0$, allow $z \rightarrow 1^-$ along the real axis, we get

$$\operatorname{Re} \frac{1 - \sum_{n=1}^{\infty} (n/\mu - 1) b_n}{(1 + \sum_{n=1}^{\infty} b_n)^2} \geq 0,$$

which gives that

$$\sum_{n=1}^{\infty} (n - \mu) b_n \leq \mu$$

and so by Lemma 5.9, we have $f \in \mathcal{U}(1, \mu)$. \square

Theorem 5.12. *Let $0 < \mu \leq 1$. A function f of the form (5.1) with $b_n \geq 0$ and $z/f(z) \neq 0$, is in $\mathcal{U}(1, \mu)$ if and only if*

$$(5.11) \quad \sum_{n=1}^{\infty} (n - \mu) b_n \leq \mu.$$

Proof. In view of Lemma 5.9, it suffices to prove the necessary part. To do this, we let $f \in \mathcal{U}(1, \mu)$ and f is of the form (5.1). Then using (5.3), we get

$$\left| \left(\frac{z}{f(z)} \right)^{\mu+1} f'(z) - 1 \right| = \left| \left(\frac{z}{f(z)} \right)^{\mu} - \frac{z}{\mu} \frac{d}{dz} \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| = \frac{1}{\mu} \left| \sum_{n=1}^{\infty} (n - \mu) b_n z^n \right| \leq 1.$$

Because $b_n \geq 0$, letting $z \rightarrow 1^-$ along the real axis, we obtain the coefficient condition (5.11). \square

The following result gives a sufficient condition for starlike functions of order α to be in the class $\mathcal{U}(\lambda, \mu)$.

Theorem 5.13. *If $f \in \mathcal{S}^*(\alpha)$ is of the form (5.1) with $b_n \geq 0$ and $\mu > 0$, then*

$$(5.12) \quad \sum_{n=1}^{\infty} (n - \mu(1 - \alpha)) b_n \leq \mu(1 - \alpha).$$

In particular, $f \in \mathcal{U}(1 - \alpha, \mu)$.

Proof. It is easy to see that

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) \geq \alpha \Leftrightarrow \left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| \leq 1.$$

Now, using this relation and the identity (5.3), we have the following

$$\begin{aligned} \left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| &= \left| \frac{-z \frac{d}{dz} \left(\frac{z}{f(z)} \right)^\mu}{2\mu(1-\alpha) \left(\frac{z}{f(z)} \right)^\mu - z \frac{d}{dz} \left(\frac{z}{f(z)} \right)^\mu} \right| \\ &= \left| \frac{-\sum_{n=1}^{\infty} n b_n z^n}{2\mu(1-\alpha) (1 + \sum_{n=1}^{\infty} b_n z^n) - \sum_{n=1}^{\infty} n b_n z^n} \right| \leq 1. \end{aligned}$$

Since $b_n \geq 0$, if $z \rightarrow 1^-$ along the real axis, we see from the last inequality that

$$\frac{\sum_{n=1}^{\infty} n b_n}{2\mu(1-\alpha) - \sum_{n=1}^{\infty} (n - 2\mu(1-\alpha)) b_n} \leq 1.$$

This gives the desired inequality (5.12).

Finally, since $n - \mu \leq n - \mu(1 - \alpha)$, we have

$$\sum_{n=1}^{\infty} (n - \mu) b_n \leq \sum_{n=1}^{\infty} (n - \mu(1 - \alpha)) b_n \leq \mu(1 - \alpha).$$

From Lemma 5.9, we conclude that $f \in \mathcal{U}(1 - \alpha, \mu)$. □

As a consequence of Theorem 5.13, we next see that $\mathcal{T}^*(\alpha) \subset \mathcal{U}(1 - \alpha)$.

Corollary 5.14. *If $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $\mathcal{S}^*(\alpha)$, then $f \in \mathcal{U}(1 - \alpha)$.*

Proof. Let $f \in \mathcal{S}^*(\alpha)$ be of the form $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. Then $z/f(z)$ is nonvanishing in the unit disk and so it can be expressed as

$$\frac{z}{f(z)} = \frac{1}{1 - |a_2|z - |a_3|z^2 - \dots} = 1 + b_1 z + b_2 z^2 + \dots,$$

where $b_n \geq 0$ for all $n \in \mathbb{N}$. Then by Theorem 5.13, $f \in \mathcal{U}(1 - \alpha)$. □

From [91], we collect the following result.

Lemma 5.15. *Let $0 \leq \lambda, \gamma \leq 1$ and $f \in \mathcal{U}(\lambda)$. Define*

$$\lambda_\gamma^* = \frac{-|f''(0)| \cos(\pi\gamma/4) + \sin(\pi\gamma/4) \sqrt{16 \cos^2(\pi\gamma/4) - |f''(0)|^2}}{2 \cos(\pi\gamma/4)}$$

and let $\lambda^{\mathcal{R}_\gamma}$ be given by the inequality

$$\sin(\pi\gamma/2) \sqrt{4 - \lambda^2} \geq (|f''(0)| + \lambda) \sqrt{4 - (|f''(0)| + \lambda)^2} + \lambda \cos(\pi\gamma/2).$$

Then

- (i) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}_\gamma$ for $0 < \lambda \leq \lambda_\gamma^*/2$,
- (ii) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_\gamma$ for $0 < \lambda \leq \lambda^{\mathcal{R}_\gamma}/2$,

where

$$\begin{aligned} \mathcal{R}_\gamma : &= \left\{ f \in \mathcal{A} : |\arg f'(z)| \leq \frac{\pi\gamma}{2} \right\} \quad \text{and} \\ \mathcal{S}_\gamma : &= \left\{ f \in \mathcal{A} : |\arg(zf'(z)/f(z))| \leq \frac{\pi\gamma}{2} \right\}. \end{aligned}$$

Using the containment results of Lemma 5.15 and Corollary 5.14, one can derive a number of interesting results. For instance, we obtain the following:

Corollary 5.16. *If $0 \leq \gamma \leq 1$ and $f(z) = z - \sum_{n=3}^{\infty} |a_n| z^n \in \mathcal{S}^*(1 - \sin \frac{\pi\gamma}{6})$, then $f \in \mathcal{R}_\gamma$. In particular, if $f''(0) = 0$, then $f \in \mathcal{S}^*(1/2)$ implies that $\operatorname{Re} f'(z) \geq 0$.*

Corollary 5.17. *If $0 \leq \gamma \leq 1$ and $f(z) = z - \sum_{n=3}^{\infty} |a_n| z^n \in \mathcal{S}^*(1 - \sin \frac{\pi\gamma}{4})$, then $f \in \mathcal{S}_\gamma$. In particular, if $f''(0) = 0$, then $f \in \mathcal{S}^*(1/2)$ implies that $|\arg(zf'(z)/f(z))| \leq \pi/3$.*

5.6. Conclusion

To present a meromorphic analog of the class $\mathcal{U}(\lambda)$, we recall, for example, the following result.

Lemma 5.18. [103, Theorem 1.2] *If $f \in \mathcal{U}(\lambda)$ and $a = |f''(0)|/2 \leq 1$, then $f \in \mathcal{S}^*(\delta)$ whenever $0 \leq \lambda \leq \lambda(\delta)$, where*

$$(5.13) \quad \lambda(\delta) = \begin{cases} \frac{\sqrt{(1-2\delta)(2-a^2-2\delta)} - a(1-2\delta)}{2(1-\delta)} & \text{if } 0 \leq \delta < \frac{1+a}{3+a}, \\ \frac{1-\delta(1+a)}{1+\delta} & \text{if } \frac{1+a}{3+a} \leq \delta < \frac{1}{1+a}. \end{cases}$$

In particular,

$$f \in \mathcal{U}(\lambda), \quad f''(0) = 0 \implies f \in \mathcal{S}^* \quad \text{whenever } 0 \leq \lambda \leq 1/\sqrt{2}.$$

Fournier and Ponnusamy [31] settled the question of sharpness of the bound for λ for which $\mathcal{U}(\lambda) \subset \mathcal{S}^*$. As a motivation for our next result, we consider the class, denoted by Σ , of all functions of the form

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}$$

that are analytic and univalent for $|\zeta| > 1$. Thus

$$F \in \Sigma \iff f \in \mathcal{S}, \quad f(z) = \frac{1}{F(1/z)} = \frac{z}{1 + \sum_{n=1}^{\infty} c_{n-1} z^n}.$$

Also, we note that

$$f'(z) \left(\frac{z}{f(z)} \right)^2 = F'(1/z) \quad \text{and} \quad \frac{zf'(z)}{f(z)} = \frac{(1/z)F'(1/z)}{F(1/z)}.$$

Consequently, for $0 \leq \lambda \leq 1$, $f \in \mathcal{U}(\lambda)$ if and only if $|F'(\zeta) - 1| \leq \lambda$ for $|\zeta| > 1$. Similarly, for $0 \leq \alpha \leq 1$, $f \in \mathcal{S}^*(\alpha)$ if and only if

$$\operatorname{Re} \left(\frac{\zeta F'(\zeta)}{F(\zeta)} \right) \geq \alpha \quad \text{for } |\zeta| > 1.$$

The class of all such functions satisfying the later condition is denoted by $\Sigma^*(\alpha)$. Thus, Lemma 5.18 takes the following form:

Theorem 5.19. *Let $F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}$ be analytic and univalent for $|\zeta| > 1$. If F satisfies the condition*

$$|F'(\zeta) - 1| \leq \lambda \quad \text{for } |\zeta| > 1$$

and $a = |-c_0| \leq 1$, then $F \in \Sigma^(\delta)$ whenever $0 < \lambda \leq \lambda(\delta)$, where $\lambda(\delta)$ is given by (5.13). In particular, for $c_0 = 0$, $F \in \Sigma^*(\delta)$ whenever $0 < \lambda \leq 1/\sqrt{2}$.*

This result may be used to generate a number of results for various subclasses of the class of meromorphic univalent functions.

CHAPTER 6

NORM ESTIMATES OF CERTAIN ANALYTIC FUNCTIONS

This chapter is devoted to the study of pre-Schwarzian norm estimates of certain subclasses of analytic functions. Section 6.1 consists of definitions and preliminary results. In Section 6.2, we collect some results to prove our main theorems. In Section 6.3, we state and prove our main results and some of their consequences. Finally, Section 6.4 concludes with a number of open problems.

Most of the results in this chapter are from the articles: **S. Ponnusamy and S.K. Sahoo** (2008) Norm estimates for convolution transforms of certain classes of analytic functions. *J. Math. Anal. Appl.* **342**, 171–180

and

R. Parvatham, S. Ponnusamy and S.K. Sahoo (2008) Norm estimate for the Bernardi integral transforms of functions defined by subordination. *Hiroshima Math. J.* **38**, 19–29.

6.1. Introduction

We refer to Chapter 1 for related definitions and notations used in this chapter. First we recall the subclass \mathcal{F}_β of \mathcal{A} defined by

$$\mathcal{F}_\beta = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3}{2}\beta, \quad z \in \mathbb{D} \right\}$$

for some $\beta > \frac{2}{3}$.

We also consider the subclasses $\mathcal{S}^*(A, B)$ and $\mathcal{K}(A, B)$ of \mathcal{A} defined by (see Janowski [67])

$$\mathcal{S}^*(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}$$

and

$$\mathcal{K}(A, B) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \right\}.$$

Here we assume that $-1 \leq B < A \leq 1$, but a relaxed restriction on A, B will be used in the last section. These classes are widely used in the literature. For $0 \leq \alpha < 1$, we observe that

$$\mathcal{S}^*(1 - 2\alpha, -1) = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{K}(1 - 2\alpha, -1) = \mathcal{K}(\alpha).$$

We note that $f \in \mathcal{S}^*(A, B)$ if and only if $J[f] \in \mathcal{K}(A, B)$, where $J[f]$ is defined by (1.11).

In addition, we estimate the pre-Schwarzian norm of functions from the subclass $\mathcal{S}^*(\alpha, \beta)$ of \mathcal{A} defined by

$$\mathcal{S}^*(\alpha, \beta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h_{\alpha, \beta}(z) \equiv \left(\frac{1 + (1 - 2\beta)z}{1 - z} \right)^\alpha \right\},$$

for $0 < \alpha \leq 1$ and $0 \leq \beta < 1$. Since functions in $\mathcal{S}^*(\alpha, \beta)$ belong to $\mathcal{S}^*(1, 0) \equiv \mathcal{S}^*$, $\mathcal{S}^*(\alpha, \beta) \subsetneq \mathcal{S}$ for $0 < \alpha \leq 1$ and $0 \leq \beta < 1$.

The class $\mathcal{S}^*(\alpha, \beta)$ has been studied by Wesołowski in [122]. With $0 < \alpha \leq 1$ and $0 < \beta < 1$, we have

$$h_{\alpha, \beta}(e^{i\theta}) = (\beta + i(1 - \beta) \cot(\theta/2))^\alpha$$

from which we easily see that the univalent function $h_{\alpha, \beta}(z)$ maps \mathbb{D} onto a convex domain bounded by the curve given by

$$w = \left(\frac{\beta}{\cos \phi} \right)^\alpha e^{i\alpha\phi}, \quad -\pi/2 < \phi < \pi/2,$$

where ϕ and θ satisfy the relation $(1 - \beta) \cot(\theta/2) = \beta \tan \phi$. In particular, functions in the class $\mathcal{SS}^*(\alpha) \equiv \mathcal{S}^*(\alpha, 0)$ are called the strongly starlike functions of order α ; equivalently, $f \in \mathcal{SS}^*(\alpha)$ if and only if $|\arg(zf'(z))/f(z)| < \pi\alpha/2$, for $z \in \mathbb{D}$. Every strongly starlike function f of order $\alpha < 1$ is bounded (see [23]). Further, this class of functions has been studied by many authors, for example by Sugawa (see [117]).

6.2. Preparatory Results

In this section, we collect some known results on starlikeness of hypergeometric functions and as a consequence we also obtain a useful result that deals with the starlikeness of the derivative of hypergeometric functions. We also need an invariance property of subordination in terms of convolution of convex functions.

The following result is a reformulated version of Ma and Minda [78, Theorem 1] (see also [73]).

Lemma 6.1. *Let $\psi \in \mathcal{H}_1$ be starlike and suppose that $g \in \mathcal{A}$ satisfies the equation*

$$1 + \frac{zg''(z)}{g'(z)} = \psi(z), \quad z \in \mathbb{D}.$$

Then for $f \in \mathcal{A}$, the condition $1 + zf''(z)/f'(z) \prec \psi(z)$ implies $f'(z) \prec g'(z)$.

Recall that in Lemma 6.1, the notation \mathcal{H}_1 is used for the class of analytic functions which take origin into 1.

From the theory of prestarlike functions (see [111, p. 61] and [112, Theorem B]), one obtains the following starlikeness criterion for hypergeometric functions.

Lemma 6.2. *Let a, b, c be real numbers with $0 \leq a \leq b \leq c$. Then the function $zF(a, b; c; z)$ is starlike of order $1 - a/2$.*

Starlikeness of functions in the form $zF(a, b; c; z)$ has also been studied by many other authors (see, for example, [75, 104] and the references therein).

Corollary 6.3. *Suppose that the real numbers b and c are related by $1 \leq b \leq c$ or $-1 < b \leq 1 \leq c$. Then $zF'(1, b; c; z)$ is starlike and hence $F(1, b; c; z)$ is convex.*

Proof. We have

$$zF'(1, b; c; z) = \frac{b}{c}zF(2, b+1; c+1; z) = \frac{b}{c}zF(b+1, 2; c+1; z).$$

The desired conclusion follows if we apply Lemma 6.2 to the two expressions on the right of the last equality. □

The following result is due to Ruscheweyh [111, Theorem 2.36, p. 86] (see also [29, Theorem 8.9, p. 254]):

Lemma 6.4. *Let $f \in \mathcal{H}$ and g be a convex function such that $f \prec g$. Then for all convex functions h , we have $h * f \prec h * g$.*

We also need the following integral representation of quotient of two hypergeometric functions which is due to Küstner [75, Theorem 1.5] (see also [27, Lemma 7]).

Lemma 6.5. *Suppose that $a, b, c \in \mathbb{R}$ satisfy $-1 \leq a \leq c$ and $0 < b \leq c$. Then there exists a Borel probability measure μ on the interval $[0, 1]$ such that*

$$\frac{F(a+1, b+1; c+1; z)}{F(a, b; c; z)} = \int_0^1 \frac{d\mu(t)}{1-tz}, \quad z \in \mathbb{D}.$$

6.3. Pre-Schwarzian Norm Estimates

In this section we mainly concentrate in estimating the pre-Schwarzian norm of functions and that of the transforms $B_{b,c}[f]$ of functions f from the subclasses defined in Section 6.1. We also present some consequences in terms of quasidisks.

In order to discuss norm estimates for the class \mathcal{F}_β , for $2/3 < \beta \leq 1$, $b > 0$ and $c > 0$, we define

$$L(\beta, b, c) = \frac{b}{c}(3\beta - 2) \sup_{0 \leq x < 1} (1 - x^2) \frac{F(3 - 3\beta, b + 1; c + 1; x)}{F(2 - 3\beta, b; c; x)}.$$

Here we present

Theorem 6.6. *Let $2/3 < \beta \leq 1$ and $f \in \mathcal{F}_\beta$. Then $\|f\| \leq 2(3\beta - 2)$. If moreover $1 \leq b \leq c$ or $0 < b \leq 1 \leq c$, then $\|B_{b,c}[f]\| \leq L(\beta, b, c)$. The bounds in both cases are sharp and the quantity $L(\beta, b, c)$ is bounded above by $2(3\beta - 2)b/c$.*

Proof. Let $f \in \mathcal{F}_\beta$. Then we have

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + (1 - 3\beta)z}{1 - z} = \phi(z), \quad z \in \mathbb{D},$$

where ϕ is clearly a convex function and therefore starlike. Let $g \in \mathcal{A}$ be such that

$$1 + \frac{zg''(z)}{g'(z)} = \frac{1 + (1 - 3\beta)z}{1 - z}, \quad z \in \mathbb{D}.$$

A simple computation shows that

$$g'(z) = (1 - z)^{3\beta-2} = F(1, 2 - 3\beta; 1; z)$$

so that

$$(6.1) \quad g(z) = \frac{1 - (1 - z)^{3\beta-1}}{3\beta - 1}.$$

By Lemma 6.1, we conclude that

$$(6.2) \quad f'(z) \prec g'(z) = (1 - z)^{3\beta-2}, \quad z \in \mathbb{D},$$

which, by the definition of subordination, implies that

$$f'(z) = (1 - w(z))^{3\beta-2}$$

for some Schwarz function $w(z)$, i.e. $w : \mathbb{D} \rightarrow \mathbb{D}$ is analytic with $w(0) = 0$. By Schwarz-Pick lemma we get

$$|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D},$$

and hence,

$$\begin{aligned} \left| \frac{f''(z)}{f'(z)} \right| &= (3\beta - 2) \left| \frac{w'(z)}{1 - w(z)} \right| \\ &\leq (3\beta - 2) \frac{1 - |w(z)|^2}{1 - |z|^2} \frac{1}{1 - |w(z)|} \\ &= (3\beta - 2) \frac{1 + |w(z)|}{1 - |z|^2} \end{aligned}$$

which gives that $\|f\| \leq 2(3\beta - 2)$ and the equality holds for the function $g \in \mathcal{F}_\beta$ defined in (6.1). Indeed, we compute that

$$\|g\| = (3\beta - 2) \sup_{|z| < 1} \frac{1 - |z|^2}{|1 - z|} = 2(3\beta - 2).$$

We now proceed to prove the second part. By Corollary 6.3, we observe that $g'(z)$ is convex in \mathbb{D} , since $2/3 < \beta \leq 1$. Furthermore, by Corollary 6.3, it follows that if b and c are related by $1 \leq b \leq c$ or $-1 < b \leq 1 \leq c$ (which holds by the hypothesis of the theorem), then the hypergeometric function $F(1, b; c; z)$ is convex. In view of (6.2) and Lemma 6.4, we also have

$$F(1, b; c; z) * f'(z) \prec F(1, b; c; z) * g'(z), \quad \text{i.e.} \quad (B_{b,c}[f])'(z) \prec (B_{b,c}[g])'(z).$$

We see that (see the proof of Proposition 6.21) $\|B_{b,c}[f]\| \leq \|B_{b,c}[g]\|$ holds. So it remains to compute the norm $\|B_{b,c}[g]\|$.

By the definition of Hadamard product we have

$$(J_{b,c}[g])'(z) = F(1, b; c; z) * F(1, 2 - 3\beta; 1; z) = F(2 - 3\beta, b; c; z).$$

In view of the representation $(B_{b,c}[g])'(z) = F(2 - 3\beta, b; c; z)$ and Lemma 6.5, we deduce that there exists a Borel probability measure μ on the interval $[0, 1]$ such that

$$\begin{aligned} \frac{(B_{b,c}[g])''(z)}{(B_{b,c}[g])'(z)} &= \frac{b}{c}(2 - 3\beta) \frac{F(3 - 3\beta, b + 1; c + 1; z)}{F(2 - 3\beta, b; c; z)} \\ &= \frac{b}{c}(2 - 3\beta) \int_0^1 \frac{d\mu(t)}{1 - tz}, \quad z \in \mathbb{D}, \end{aligned}$$

whenever $0 < b \leq c$ and $\frac{2-c}{3} \leq \beta \leq 1$ (and so is by the hypothesis). The above formulation clearly shows that

$$\begin{aligned} \|B_{b,c}[g]\| &= \sup_{|z| < 1} (1 - |z|^2) \left| \frac{(B_{b,c}[g])''(z)}{(B_{b,c}[g])'(z)} \right| \\ &= \sup_{0 \leq x < 1} (1 - x^2) \frac{(B_{b,c}[g])''(x)}{(B_{b,c}[g])'(x)} \\ &= \frac{b(3\beta - 2)}{c} \sup_{0 \leq x < 1} (1 - x^2) \frac{F(3 - 3\beta, b + 1; c + 1; x)}{F(2 - 3\beta, b; c; x)} \\ &= L(\beta, b, c). \end{aligned}$$

Thus, we have the sharp inequality $\|B_{b,c}[f]\| \leq L(\beta, b, c)$. To obtain an upper bound for the quantity $L(\beta, b, c)$, it suffices to observe that

$$(1 - x^2) \frac{F(3 - 3\beta, b + 1; c + 1; x)}{F(2 - 3\beta, b; c; x)} = \int_0^1 \frac{1 - x^2}{1 - tx} d\mu(t) \leq \int_0^1 (1 + x) d\mu(t) \leq 2$$

which shows that

$$L(\beta, b, c) \leq \frac{2b(3\beta - 2)}{c}.$$

We thus completed our proof. □

Recall that a quasidisk is the image of a disk under a quasiconformal self map of $\overline{\mathbb{C}}$. In 1984 the following theorem was proved by Becker and Pommerenke [11] (see also [47, Theorem 4.2]).

Theorem 6.7. *If $\|f\| < 1$, then $f(\mathbb{D})$ is a quasidisk.*

As a consequence of Theorem 6.6, by using Theorem 6.7, we obtain the following.

Corollary 6.8. *For $f \in \mathcal{F}_\beta$, we obtain that $f(\mathbb{D})$ is a quasidisk if $2/3 < \beta < 5/6$.*

In the case $\beta = 1$, Theorem 6.6 takes the following simple form.

Corollary 6.9. *Suppose that $1 \leq b \leq c$ or $0 < b \leq 1 \leq c$ holds. If $f \in \mathcal{F}$, then we have $\|f\| \leq 2$ and*

$$\|B_{b,c}[f]\| \leq L(1, b, c) = \frac{2(c - \sqrt{c^2 - b^2})}{b}.$$

The bounds are sharp.

Proof. From Theorem 6.6, we see that

$$L(1, b, c) = \frac{b}{c} \sup_{0 \leq x < 1} (1 - x^2) \frac{F(0, b + 1; c + 1; x)}{F(-1, b; c; x)} = \frac{b}{c} \sup_{0 \leq x < 1} \frac{1 - x^2}{1 - (b/c)x}.$$

For $b = c > 0$ the conclusion is obvious. For $b/c < 1$ ($b > 0, c > 0$), it is a simple exercise to see that the function $h(x) = (1 - x^2)/(1 - (b/c)x)$ defined on $[0, 1)$ attains its maximum at

$$x_0 = \frac{c - \sqrt{c^2 - b^2}}{b}$$

so that

$$h(x) \leq h(x_0) = \frac{2c(c - \sqrt{c^2 - b^2})}{b^2}$$

and the conclusion follows. □

Here we see that

$$\frac{2(c - \sqrt{c^2 - b^2})}{b} \leq 1 \iff b \leq \frac{4}{5}c.$$

Thus, as a consequence of Corollary 6.9 we obtain the following by using a result of Becker [10] and Theorem 6.7.

Corollary 6.10. *Let f be in \mathcal{F} . Then $J_{b,c}[f]$ is univalent if $1 \leq b \leq \frac{4}{5}c$ and $J_{b,c}[f](\mathbb{D})$ is a quasidisk if $1 \leq b < \frac{4}{5}c$.*

As a consequence of Corollary 6.9, we easily have

Corollary 6.11. *Let $\gamma > -1$, and $B_\gamma[f]$ be the Bernardi transform of $f \in \mathcal{F}$. Then, we have*

$$\|B_\gamma[f]\| \leq \frac{2(\gamma + 2 - \sqrt{3 + 2\gamma})}{\gamma + 1}$$

and the bound is sharp.

Here we compute that

$$\frac{2(\gamma + 2 - \sqrt{3 + 2\gamma})}{\gamma + 1} \leq 1 \iff \gamma \leq 3,$$

because $\gamma + 1 > 0$. Thus, as a consequence of Corollary 6.11 we have the following result.

Corollary 6.12. *Let f be in \mathcal{F} . Then $B_\gamma[f]$ is univalent for $-1 < \gamma \leq 3$ and $B_\gamma[f](\mathbb{D})$ is a quasidisk for $-1 < \gamma < 3$.*

Setting $\gamma = 0$ and $\gamma = 1$ respectively. Thus, we have

Corollary 6.13. *Let $f \in \mathcal{F}$. Then we have $\|J[f]\| \leq 4 - 2\sqrt{3}$ and $\|L[f]\| \leq 3 - \sqrt{5}$. The bounds are sharp.*

Combining Theorem 6.7 and Corollary 6.13, we obtain the following.

Corollary 6.14. *If $f \in \mathcal{F}$, then the images $J[f](\mathbb{D})$ and $L[f](\mathbb{D})$ are quasidisks.*

The class \mathcal{F} is particularly interesting because of the inclusion $\mathcal{F} \subset \mathcal{S}^* \subset \mathcal{S}$. On the other hand, if $f \in \mathcal{S}^*$, then $\|f\| \leq 6$ and $\|J[f]\| \leq 4$. Both the bounds here are sharp and was proved by S. Yamashita [125] (see also [27, Theorem A]). Later from Corollary 6.17, we see that if $f \in \mathcal{K}$, then $\|f\| \leq 4$, $\|J[f]\| \leq 2$ and $\|L[f]\| \leq 8/3$. All these bounds are sharp.

Corresponding to the class $\mathcal{K}(A, B)$, $-1 \leq B < A \leq 1$, we introduce $N(A, B)$

$$(6.3) \quad N(A, B) := \begin{cases} 2(A - B) \left[\frac{1 - \sqrt{1 - B^2}}{B^2} \right] & \text{for } B \neq 0, \\ A & \text{for } B = 0. \end{cases}$$

To state our next theorem, we also need to define another quantity $M(A, B, b, c)$ by

$$(6.4) \quad M(A, B, b, c) := \frac{b(A - B)}{c} \sup_{0 \leq x < 1} (1 - x^2) \frac{F(2 - A/B, b + 1; c + 1; |B|x)}{F(1 - A/B, b; c; |B|x)}$$

where A, B, b, c are related by

$$(6.5) \quad -1 \leq B < A \leq \min\{1, B+1\}, \quad B \neq 0, \quad 1 \leq b \leq c, \quad \text{and} \quad -2 \leq -A/B \leq c-1$$

or

$$(6.6) \quad -1 \leq B < A \leq \min\{1, B+1\}, \quad B \neq 0, \quad 0 < b \leq 1 \leq c, \quad \text{and} \quad -2 \leq -A/B \leq c-1.$$

Theorem 6.15. *Let $-1 \leq B < A \leq 1$ and $f \in \mathcal{K}(A, B)$. Then $\|f\| \leq N(A, B)$. If moreover the real constants A, B, b, c are related by (6.5) or (6.6), then $\|B_{b,c}[f]\| \leq M(A, B, b, c)$. The bounds are sharp and the quantity $M(A, B, b, c)$ is bounded from above by $\frac{b}{c}(1 + |B|)(A - B)$.*

Proof. The proof is similar to that of Theorem 6.6. Suppose that $f \in \mathcal{K}(A, B)$. In terms of subordination, f can be characterized by

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} = \phi_{A,B}(z), \quad z \in \mathbb{D},$$

where $\phi_{A,B}$ is known to be a convex function and therefore starlike. Define $g \in \mathcal{A}$ by the relation

$$(6.7) \quad 1 + \frac{zg''(z)}{g'(z)} = \frac{1 + Az}{1 + Bz}, \quad z \in \mathbb{D}.$$

By Lemma 6.1, we have

$$(6.8) \quad f'(z) \prec g'(z) = \begin{cases} (1 + Bz)^{(A/B)-1} & \text{if } B \neq 0, \\ e^{Az} & \text{if } B = 0. \end{cases}$$

If $B = 0$, then we see that $f'(z) \prec e^{Az}$ for $0 < |A| \leq 1$ and so, by the definition of subordination, we have $f'(z) = e^{Aw(z)}$ for some Schwarz function $w(z)$. By Schwarz-Pick lemma we obtain

$$(1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| \leq |A|(1 - |w(z)|^2), \quad z \in \mathbb{D},$$

and hence, for $B = 0$ and $0 < |A| \leq 1$, we finally get $\|f\| \leq |A|$. The estimate is sharp for the function $f(z) = (e^{Az} - 1)/A$.

On the other hand, if $0 \neq B$ and $-1 \leq B < A \leq 1$, then by the same process we see that

$$(1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| \leq \frac{(A - B)(1 - |w(z)|^2)}{1 - |B||w(z)|}$$

for some Schwarz function $w(z)$ and hence we obtain

$$\|f\| \leq (A - B) \sup_{0 \leq x < 1} \frac{1 - x^2}{1 - |B|x} = 2(A - B) \left[\frac{1 - \sqrt{1 - B^2}}{B^2} \right].$$

Thus, for $-1 \leq B < A \leq 1$, we formulate the pre-Schwarzian norm estimates of the functions $f \in \mathcal{K}(A, B)$ by $\|f\| \leq N(A, B)$, where $N(A, B)$ is defined by (6.3).

Our next task is to show that

$$\|B_{b,c}[f]\| \leq \|B_{b,c}[g]\|.$$

To do this, we first observe the fact that $f'(z) \prec g'(z)$ in \mathbb{D} . The convexity of $g'(z)$ is easy when $A \leq B + 1 \neq 1$. Indeed, set $h = g'$. By the defining relation (6.7) we then have

$$\frac{h'(z)}{h(z)} = \frac{A - B}{1 + Bz}.$$

Taking the logarithmic derivative of both sides and multiplying with z , we obtain

$$\frac{zh''(z)}{h'(z)} - \frac{zh'(z)}{h(z)} = -\frac{Bz}{1 + Bz}.$$

Therefore,

$$1 + \frac{zh''(z)}{h'(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{Bz}{1 + Bz} = \frac{1 + Az}{1 + Bz} - \frac{Bz}{1 + Bz} = \frac{1 + (A - B)z}{1 + Bz}.$$

We write

$$S(z) = \frac{1 + (A - B)z}{1 + Bz}, \quad z \in \mathbb{D}.$$

Since the Möbius transformation $S(z)$ has no pole in the unit disk \mathbb{D} , the image $S(\mathbb{D})$ is the disk centered at $\frac{1 - B(A - B)}{1 - B^2}$ and radius $\frac{A - 2B}{1 - B^2}$. Clearly the points $S(-1)$ and $S(1)$ are diametrically opposite points to this disk. Therefore, $h(z)$ is convex (equivalently, $S(z) = 1 + zh''(z)/h'(z)$ has a positive real part) if and only if $S(-1) \geq 0$ and $S(1) \geq 0$. The last condition is equivalent to $A \leq B + 1$. This shows that $g'(z)$ is convex for $A \leq B + 1 \neq 1$.

Also, Corollary 6.3 says that if b and c are related by $1 \leq b \leq c$ or $-1 < b \leq 1 \leq c$, then $F(1, b; c; z)$ is convex. Consequently, as in the proof of Theorem 6.6, Lemma 6.4 gives

$$(B_{b,c}[f])'(z) = F(1, b; c; z) * f'(z) \prec (B_{b,c}[g])'(z) = F(1, b; c; z) * g'(z)$$

whenever $A \leq B + 1$ and b, c satisfy by $1 \leq b \leq c$ or $-1 < b \leq 1 \leq c$. Thus,

$$\|B_{b,c}[f]\| \leq \|B_{b,c}[g]\|$$

holds.

Finally, it remains to compute the norm $\|B_{b,c}[g]\|$ for $B \neq 0$. Since

$$(1 + Bz)^{(A/B)-1} = F(1, 1 - A/B; 1; -Bz) \quad \text{for } B \neq 0,$$

it follows from the definition of the hypergeometric function that

$$(B_{b,c}[g])'(z) = F(1, b; c; z) * F(1, 1 - A/B; 1; -Bz) = F(1 - A/B, b; c; -Bz)$$

and so we can write

$$\frac{(B_{b,c}[g])''(z)}{(B_{b,c}[g])'(z)} = \frac{b(A - B)}{c} \frac{F(2 - A/B, b + 1; c + 1; -Bz)}{F(1 - A/B, b; c; -Bz)}.$$

If $0 < |B| \leq 1$, then by Lemma 6.5 we can easily obtain

$$\|B_{b,c}[g]\| = M(A, B, b, c)$$

whenever $0 < b \leq c$, $B < A$ and $-2 \leq -A/B \leq c - 1$, where $M(A, B, b, c)$ is defined by (6.4). This proves the sharpness of the norm estimate of $\|B_{b,c}[f]\|$ whenever (6.5) or (6.6) holds.

Finally, we establish an upper bound for the quantity $M(A, B, b, c)$. Again, using Lemma 6.5, we may express

$$(1 - x^2) \frac{F(2 - A/B, b + 1; c + 1; |B|x)}{F(1 - A/B, b; c; |B|x)} = \int_0^1 \frac{1 - x^2}{1 - t|B|x} d\mu(t)$$

for some Borel probability measure μ on the interval $[0, 1]$ and under the hypotheses on the constants A, B, b, c . Since

$$\frac{1 - x^2}{1 - t|B|x} \leq \frac{1 - |B|^2 x^2}{1 - |B|x} = 1 + |B|x \leq 1 + |B| \quad \text{for } 0 \leq t \leq 1,$$

the inequality

$$(1 - x^2) \frac{F(2 - A/B, b + 1; c + 1; |B|x)}{F(1 - A/B, b; c; |B|x)} \leq 1 + |B|x \leq 1 + |B|$$

holds for $0 \leq x < 1$. This gives that

$$M(A, B, b, c) \leq \frac{b}{c} (1 + |B|)(A - B)$$

and we complete the proof. \square

Remark 6.16. In the proof of Theorem 6.15, we have established the pre-Schwarzian norm estimate of $f \in \mathcal{K}(A, 0)$ although this is not stated in the statement. However, we do not have an answer in finding norm estimate for $B_{b,c}[f]$ when $f \in \mathcal{K}(A, 0)$.

If one chooses $c = b + 1 = \gamma + 2$, then we obtain that

$$\begin{aligned} D(A, B, \gamma) &:= (A - B) \left(\frac{\gamma + 1}{\gamma + 2} \right) \sup_{0 \leq x < 1} (1 - x^2) \frac{F(2 - A/B, \gamma + 2; \gamma + 3; |B|x)}{F(1 - A/B, \gamma + 1; \gamma + 2; |B|x)} \\ &= M(A, B, \gamma + 1, \gamma + 2), \end{aligned}$$

where A, B, γ are related by

$$(6.9) \quad -1 \leq B < A \leq \min\{1, B + 1\}, \quad B \neq 0, \quad -1 < \gamma \text{ and } -2 \leq -A/B \leq \gamma + 1.$$

Thus, Theorem 6.15 leads to the following result.

Theorem 6.17. *Let A, B, γ be real constants satisfying the condition (6.9). Then for every $f \in \mathcal{K}(A, B)$, the Bernardi transform $B_\gamma[f]$ of f satisfies the inequality $\|B_\gamma[f]\| \leq D(A, B, \gamma)$. The bound $D(A, B, \gamma)$ is sharp and satisfies*

$$D(A, B, \gamma) \leq \frac{(1 + |B|)(A - B)(\gamma + 1)}{\gamma + 2}.$$

Theorem 6.17 actually extends the recent work in [27]. We remark that

$$N(1, -1) = 4, \quad D(1, -1, 0) = 2, \quad \text{and} \quad D(1, -1, 1) = 8/3.$$

For the special case $B = -A$, Theorem 6.17 yields the following simple result:

Corollary 6.18. *Let $0 < A \leq 1$ and $\gamma \geq 0$. We have then*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 - Az} \implies \|B_\gamma[f]\| \leq D(A, -A, \gamma).$$

The bound $D(A, -A, \gamma)$ is sharp and satisfies

$$D(A, -A, \gamma) \leq \frac{2A(1 + A)(\gamma + 1)}{\gamma + 2}.$$

Remark 6.19. We have proved Theorem 6.17 in the article “**R. Parvatham, S. Pon-nusamy and S.K. Sahoo.** Norm estimate for the Bernardi integral transforms of functions defined by subordination. *Hiroshima Math. J.* (to appear)” separately, but not using the exact method that we use in the proof of Theorem 6.15. Indeed, we mention that Theorem 6.17 has been obtained by proving the following proposition. Because of independent interest, we describe the proposition in detail.

Note that we have used following two lemmas. One is Lemma 6.1 and the second one, due to Hallenbeck and Ruscheweyh [48], is stated below.

Lemma 6.20. [48] *Let $p(z)$ and $q(z)$ be analytic functions in the unit disk \mathbb{D} with $p(0) = 1 = q(0)$. For $\alpha > 0$ suppose that the function $h(z) = q(z) + \alpha z q'(z)$ is convex. Then the condition $p(z) + \alpha z p'(z) \prec h(z)$ implies $p(z) \prec q(z)$.*

Proposition 6.21. *Let $\gamma > -1$ be given. Suppose that the function $\psi(z) = 1 + z g''(z)/g'(z)$ is starlike and that the function $g'(z)$ is convex for a given function $g \in \mathcal{A}$. If a function $f \in \mathcal{A}$ satisfies*

$$1 + \frac{z f''(z)}{f'(z)} \prec \psi(z), \quad z \in \mathbb{D}$$

then the inequalities $\|f\| \leq \|g\|$ and $\|B_\gamma[f]\| \leq \|B_\gamma[g]\|$ hold.

Proof. First, by Lemma 6.1, the hypothesis implies that $f'(z) \prec g'(z)$. Namely, $f'(z) = (g' \circ \omega)(z)$ for some Schwarz function ω . By the Schwarz-Pick lemma, we have the inequality

$$\frac{|\omega'(z)|}{1 - |\omega|^2} \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Since a logarithmic differentiation yields $f''/f' = (g''/g') \circ \omega \cdot \omega'$, we compute

$$(1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right| = (1 - |z|^2) |\omega'(z)| \left| \frac{g''(\omega(z))}{g'(\omega(z))} \right| \leq (1 - |\omega(z)|^2) \left| \frac{g''(\omega(z))}{g'(\omega(z))} \right| \leq \|g\|.$$

Therefore, we obtain the inequality $\|f\| \leq \|g\|$. Now we proceed to prove the inequality $\|B_\gamma[f]\| \leq \|B_\gamma[g]\|$. It is enough to prove that $(B_\gamma[f])'(z) \prec (B_\gamma[g])'(z)$. It is easy to see that the Bernardi transform $B_\gamma[g]$ of g defined by (1.13) satisfies the equation

$$z(B_\gamma[g])'(z) + \gamma B_\gamma[g](z) = (\gamma + 1)g(z)$$

and so,

$$z(B_\gamma[g])''(z) + (\gamma + 1)(B_\gamma[g])'(z) = (\gamma + 1)g'(z).$$

In a similar fashion, we have

$$z(B_\gamma[f])''(z) + (\gamma + 1)(B_\gamma[f])'(z) = (\gamma + 1)f'(z).$$

Set $p(z) = (B_\gamma[f])'(z)$ and $q(z) = (B_\gamma[g])'(z)$. Then, the condition $f'(z) \prec g'(z)$ is equivalent to

$$zp'(z) + (\gamma + 1)p(z) = (\gamma + 1)f'(z) \prec (\gamma + 1)g'(z) = zq'(z) + (\gamma + 1)q(z).$$

This shows that

$$\frac{zp'(z)}{\gamma + 1} + p(z) \prec \frac{zq'(z)}{\gamma + 1} + q(z), \quad z \in \mathbb{D}.$$

Since $g'(z)$ is convex, by Lemma 6.20, we get

$$(B_\gamma[f])'(z) = p(z) \prec q(z) = (B_\gamma[g])'(z)$$

for $\gamma > -1$. We thus proved the required inequality. \square

Next, we are going to discuss the norm estimates for the class $\mathcal{S}^*(\alpha, \beta)$ defined in Section 6.1. Recall that in [117], Sugawa has presented the sharp norm estimates for functions $f \in \mathcal{SS}^*(\alpha)$. The aim of the last result of this chapter is to generalize the result of Sugawa [117, Theorem 1.1]. But unfortunately, we do not have an sharp norm estimate although we have an optimal estimate in the following form:

Theorem 6.22. *Let $0 < \alpha < 1$ and $0 \leq \beta < 1$. If $f \in \mathcal{S}^*(\alpha, \beta)$, then*

$$(6.10) \quad \|f\| \leq L(\alpha, \beta) + 2\alpha,$$

where

$$(6.11) \quad L(\alpha, \beta) = \frac{4(1 - \beta)(k - \beta)(k^\alpha - 1)}{(k - 1)(k + 1 - 2\beta)}$$

and k is the unique solution of the following equation in $x \in (1, \infty)$:

$$(6.12) \quad (1 - \alpha)x^{\alpha+2} + \beta(3\alpha - 2)x^{\alpha+1} + [(1 - 2\beta)(1 + \alpha) + 2\beta^2(1 - \alpha)]x^\alpha - \alpha\beta(1 - 2\beta)x^{\alpha-1} - x^2 + 2\beta x = (1 - \beta)^2 + \beta^2.$$

Remark 6.23. For $\alpha = 1$, it is well known that $\|f\| \leq 6 - 4\beta$ and equality holds if and only if $f(z) = \bar{\mu}\Phi(\mu z)$, where $\Phi(z) = z/(1 - z)^{2(1-\beta)}$ and μ is a unimodular constant (see [125]). Moreover, if $\alpha = 1$ as well as $\beta = 0$, it is known that $\|f\| \leq 6$; and equality holds

for the Koebe function $k(z) = z/(1-z)^2$. Now we shall prove the main theorem by using the method adopted by Sugawa [117].

Proof. Let $p(z) = P_f(z) = zf'(z)/f(z)$ and f belong to the class $\mathcal{S}^*(\alpha, \beta)$. Then, by the definition, $p(z)$ is subordinate to the univalent function

$$q(z) = \left(\frac{1 + (1 - 2\beta)z}{1 - z} \right)^\alpha, \quad z \in \mathbb{D},$$

and therefore, there exists an analytic function $\omega : \mathbb{D} \rightarrow \mathbb{D}$ with $\omega(0) = 0$ such that

$$(6.13) \quad p = q \circ \omega = \left(\frac{1 + (1 - 2\beta)\omega}{1 - \omega} \right)^\alpha.$$

Let $F \in \mathcal{A}$ be the function with $P_F = q$, i.e.

$$F(z) = z \exp \left(\int_0^z \frac{q(t) - 1}{t} dt \right).$$

We split the proof into two cases. Assume first that $0 \leq \beta \leq 1/2$. Logarithmic differentiation of (6.13) yields that

$$1 + \frac{zf''}{f'} - \frac{zf'}{f} = \frac{2\alpha(1 - \beta)z\omega'}{(1 - \omega)(1 + (1 - 2\beta)\omega)}.$$

We thus have

$$(6.14) \quad T_f(z) = \frac{2\alpha(1 - \beta)\omega'(z)}{(1 - \omega(z))(1 + (1 - 2\beta)\omega(z))} + \frac{p(z) - 1}{z}.$$

By triangle inequality and Schwarz-Pick lemma, we obtain

$$\begin{aligned} |T_f(z)| &\leq \frac{2\alpha(1 - \beta)|\omega'(z)|}{|1 - 2\beta\omega(z) - (1 - 2\beta)\omega^2(z)|} + \frac{|p(z) - 1|}{|z|} \\ &\leq \frac{2\alpha(1 - \beta)(1 - |\omega(z)|^2)}{(1 - |z|^2)(|1 - 2\beta\omega(z)| - (1 - 2\beta)|\omega(z)|^2)} + \frac{|q(\omega(z)) - 1|}{|z|} \\ &\leq \frac{2\alpha(1 - \beta)(1 - |\omega(z)|^2)}{(1 - |z|^2)(1 - 2\beta|\omega(z)| - (1 - 2\beta)|\omega(z)|^2)} + \frac{|q(\omega(z)) - 1|}{|z|} \\ &\leq \frac{2\alpha(1 - \beta)(1 + |\omega(z)|)}{(1 - |z|^2)(1 + (1 - 2\beta)|\omega(z)|)} + \frac{|q(\omega(z)) - 1|}{|z|}. \end{aligned}$$

Using a similar argument, namely the triangle inequality (as we did in the denominator above), we see that

$$\begin{aligned}
|q(z) - 1| &= \left| \int_0^z q'(t) dt \right| \\
&= \left| \int_0^z \left(\frac{1 + (1 - 2\beta)t}{1 - t} \right)^\alpha \frac{2\alpha(1 - \beta)}{(1 - t)(1 + (1 - 2\beta)t)} dt \right| \\
&\leq \int_0^{|z|} \left(\frac{1 + (1 - 2\beta)t}{1 - t} \right)^\alpha \frac{2\alpha(1 - \beta)}{(1 - t)(1 + (1 - 2\beta)t)} dt \\
&= q(|z|) - 1.
\end{aligned}$$

So, using this inequality and the fact $|\omega(z)| \leq |z|$, we get

$$\begin{aligned}
|T_f(z)| &\leq \frac{2\alpha(1 - \beta)(1 + |\omega(z)|)}{(1 - |z|^2)(1 + (1 - 2\beta)|\omega(z)|)} + \frac{q(|\omega(z)|) - 1}{|z|} \\
&\leq \frac{2\alpha(1 - \beta)(1 + |z|)}{(1 - |z|^2)(1 + (1 - 2\beta)|z|)} + \frac{q(|z|) - 1}{|z|} \\
&= T_F(|z|),
\end{aligned}$$

where the second inequality is strict provided $\omega(z)/z$ is not a unimodular constant. Therefore, we see that $\|f\| \leq \|F\|$.

Since

$$(1 - t^2)T_F(t) = \frac{2\alpha(1 - \beta)(1 + t)}{1 + (1 - 2\beta)t} + \frac{1 - t^2}{t}(q(t) - 1) \rightarrow 2\alpha \text{ as } t \rightarrow 1^-,$$

the equality $\|f\| = \|F\|$ holds only if $|T_f(z_0)| = T_F(|z_0|)$ for some $z_0 \in \mathbb{D}$. Hence we conclude that equality holds if $P_f(z) = q(\mu z)$ for some unimodular constant μ .

We next consider the case $1/2 \leq \beta < 1$. If we use triangle inequality again without multiplying the factors in the denominator, we obtain

$$|q(z) - 1| \leq q(|z|) - 1.$$

Now using the same argument as in the first case, we get

$$\begin{aligned}
(1 - |z|^2)|T_f(z)| &\leq \frac{2\alpha(1 - \beta)(1 - |\omega^2(z)|)}{|1 - \omega(z)||1 + (1 - 2\beta)\omega(z)|} + \frac{1 - |z|^2}{|z|}(q(|\omega(z)|) - 1) \\
&\leq \frac{2\alpha(1 - \beta)(1 + |\omega(z)|)}{1 + (1 - 2\beta)|\omega(z)|} + \frac{1 - |z|^2}{|z|}(q(|\omega(z)|) - 1) \\
&\leq \frac{2\alpha(1 - \beta)(1 + |z|)}{1 + (1 - 2\beta)|z|} + \frac{1 - |z|^2}{|z|}(q(|z|) - 1) \\
&= (1 - |z|^2)T_F(|z|).
\end{aligned}$$

This shows that $\|f\| \leq \|F\|$ and the inequality is sharp (as in the argument of the previous case). Thus, it is enough to compute $\|F\|$. Now, we write

$$L(\alpha, \beta) = \sup_{0 < t < 1} \frac{1 - t^2}{t}(q(t) - 1) = \sup_{x > 1} g(x),$$

where

$$g(x) = \frac{4(1 - \beta)(x - \beta)(x^\alpha - 1)}{(x - 1)(x + 1 - 2\beta)}$$

with the substitution $x = [1 + (1 - 2\beta)t]/(1 - t)$. Logarithmic derivative of $g(x)$ yields

$$\frac{g'(x)}{g(x)} = -\frac{h(x)}{(x - \beta)(x^\alpha - 1)(x - 1)(x + 1 - 2\beta)},$$

where $h(x)$ is given by

$$\begin{aligned}
h(x) &= (1 - \alpha)x^{\alpha+2} + \beta(3\alpha - 2)x^{\alpha+1} + [(1 + \alpha)(1 - 2\beta) + 2\beta^2(1 - \alpha)]x^\alpha \\
&\quad - \alpha\beta(1 - 2\beta)x^{\alpha-1} - x^2 + 2\beta x - (1 - \beta)^2 - \beta^2.
\end{aligned}$$

Differentiations give easily the following:

$$\begin{aligned}
h'(x) &= (1 - \alpha)(\alpha + 2)x^{\alpha+1} + \beta(3\alpha - 2)(\alpha + 1)x^\alpha + \alpha[(1 + \alpha)(1 - 2\beta) + 2\beta^2(1 - \alpha)]x^{\alpha-1} \\
&\quad - \alpha\beta(\alpha - 1)(1 - 2\beta)x^{\alpha-2} - 2x + 2\beta \\
h''(x) &= (1 - \alpha)(\alpha + 2)(\alpha + 1)x^\alpha + \alpha\beta(3\alpha - 2)(\alpha + 1)x^{\alpha-1} + \alpha(\alpha - 1)[(1 + \alpha)(1 - 2\beta) \\
&\quad + 2\beta^2(1 - \alpha)]x^{\alpha-2} - \alpha\beta(\alpha - 1)(\alpha - 2)(1 - 2\beta)x^{\alpha-3} - 2 \\
h'''(x) &= (1 - \alpha)(\alpha + 1)(\alpha + 2)\alpha x^{\alpha-1} + \alpha\beta(3\alpha - 2)(\alpha + 1)(\alpha - 1)x^{\alpha-2} + \alpha(\alpha - 1)(\alpha - 2) \\
&\quad [(1 + \alpha)(1 - 2\beta) + 2\beta^2(1 - \alpha)]x^{\alpha-3} - \alpha\beta(\alpha - 1)(\alpha - 2)(\alpha - 3)(1 - 2\beta)x^{\alpha-4} \\
&= \alpha(1 - \alpha)x^{\alpha-4}\phi(x)
\end{aligned}$$

where

$$\begin{aligned}\phi(x) = & (\alpha + 1)(\alpha + 2)x^3 - \beta(3\alpha - 2)(\alpha + 1)x^2 - (\alpha - 2)[(1 + \alpha)(1 - 2\beta) + 2\beta^2(1 - \alpha)]x \\ & + \beta(1 - 2\beta)(\alpha - 2)(\alpha - 3).\end{aligned}$$

It follows that

$$\phi'(x) = 3(\alpha + 1)(\alpha + 2)x^2 + 2\beta(2 - 3\alpha)(1 + \alpha)x + (2 - \alpha)[(1 + \alpha)(1 - 2\beta) + 2\beta^2(1 - \alpha)]$$

and

$$\phi''(x) = 6(\alpha + 1)(\alpha + 2)x + 2\beta(2 - 3\alpha)(1 + \alpha).$$

Since $\phi'''(x) = 6(\alpha + 1)(\alpha + 2) > 0$, $\phi''(x)$ is increasing for all $x > 1$. So we have

$$\phi''(x) \geq \phi''(1) = 6\alpha^2(1 - \beta) + 16\alpha + 12 + 4\beta + 2\alpha(1 - \beta) > 0.$$

This implies that $\phi'(x)$ is increasing for $x > 1$ and so

$$\phi'(x) \geq \phi'(1) = 2(1 + \alpha)(\alpha + 2 + 2(1 - \alpha\beta)) + 2\beta^2(1 - \alpha)(2 - \alpha) > 0.$$

So $\phi(x)$ is also increasing for $x > 1$ and hence,

$$\phi(x) \geq \phi(1) = 4(1 - \beta)(1 + \alpha + \beta + \beta(1 - \alpha)) > 0.$$

Therefore, $h'''(x) > 0$ and so $h''(x)$ is increasing for $x > 1$. Since $h''(x)$ is increasing in $(1, \infty)$ and

$$h''(1) = -2\alpha(1 - \beta)[\alpha(1 - \beta) + \beta] < 0,$$

we see that $h''(x)$ has a unique zero in $(1, \infty)$, say $x = x_1$. Since $h'(1) = 0$ and $h'(x)$ is increasing on (x_1, ∞) and decreasing on $(1, x_1)$, we obtain that $h'(x)$ has a unique zero, say x_2 ($x_2 > x_1$) in $(1, \infty)$. Since $h(1) = 0$, by the same argument we conclude that $h(x)$ has a unique zero, say $k = k(\alpha, \beta) > x_2$ in $(1, \infty)$. Thus $h(x) < 0$ in $(1, k)$ and $h(x) > 0$ in (k, ∞) , equivalently, $g'(x)$ is positive for $x \in (1, k)$ and negative for $x > k$. This shows that $g(x)$ assumes its maximum at $x = k$ and hence we have (6.11). Since k is the zero of $h(x)$, it is the unique solution of the equation (6.12). Thus we have established (6.10). \square

Remark 6.24. Here we calculate some bounds for $L(\alpha, \beta)$ and $k(\alpha, \beta)$ although these are not better estimates. Since $g(x)$ attains its maximum at $k > 1$, we note that

$$L(\alpha, \beta) = g(k) > \lim_{x \rightarrow 1^+} g(x) = 2\alpha(1 - \beta).$$

Finally we observe that, $g(x)$ satisfies the second order differential equation

$$A(x)g''(x) + B(x)g'(x) + C(x)g(x) = 0$$

where

$$\begin{aligned} A(x) &= x(x-1)(x+1-2\beta)(x-\beta)^2 \\ B(x) &= 4x(x-\beta)^3 + (1-\alpha)(x-1)(x+1-2\beta)(x-\beta)^2 \\ &\quad - 2x(x-1)(x+1-2\beta)(x-\beta) \\ C(x) &= 2(1-\alpha)(x-\beta)^3 - 2x(x-\beta)^2 - (1-\alpha)(x-1)(x+1-2\beta)(x-\beta) \\ &\quad + 2x(x-1)(x+1-2\beta). \end{aligned}$$

This observation is perhaps to justify its close connection between these bounds and special functions.

6.4. Concluding Remarks

Let β, γ, A and B be real numbers and suppose that $\beta > 0$, $\beta + \gamma > 0$, $-1 \leq B < 1$ and $B < A \leq 1 + \gamma(1-B)\beta^{-1}$. For $f \in \mathcal{S}^*(A, B)$, we consider $g = J_{\beta, \gamma}[f]$ defined by

$$(6.15) \quad g(z) = J_{\beta, \gamma}[f](z) = \left[\frac{\beta + \gamma}{z^\gamma} \int_0^z t^{\gamma-1} f^\beta(t) dt \right]^{1/\beta}, \quad z \in \mathbb{D}.$$

Moreover, we define the order of (univalent) starlikeness of the class $J_{\beta, \gamma}[\mathcal{S}^*(A, B)]$ by the largest number $\delta = \delta(A, B; \beta, \gamma)$ such that

$$J_{\beta, \gamma}[\mathcal{S}^*(A, B)] \subset \mathcal{S}^*(\delta).$$

Before we propose a general problem, we recall a special case of a result from [96].

Lemma 6.25. *Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral operator defined by (6.15).*

- (a) *If $-1 \leq B < 1$ and $B < A \leq 1 + \gamma(1-B)\beta^{-1}$, then the order of (univalent) starlikeness of $J_{\beta, \gamma}[\mathcal{S}^*(A, B)]$ is given by*

$$\delta(A, B; \beta, \gamma) = \inf_{|z| < 1} \operatorname{Re} q(z),$$

where q is given by

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}$$

with

$$Q(z) = \begin{cases} \int_0^1 \left(\frac{1+Bzt}{1+Bz} \right)^{\beta((A-B)/B)} t^{\beta+\gamma-1} dt & \text{if } B \neq 0, \\ \int_0^1 t^{\beta+\gamma-1} \exp(\beta Az(t-1)) dt & \text{if } B = 0 \end{cases}$$

and

$$q(z) = \frac{\beta - \gamma Bz}{\beta(1+Bz)} \quad \text{when } A = -\frac{(\gamma+1)B}{\beta}, \quad B \neq 0.$$

(b) Moreover, if $-1 \leq B < 0$, $B < A \leq \min\{1 + \gamma(1-B)\beta^{-1}, -(\gamma+1)B\beta^{-1}\}$, then

$$(6.16) \quad \delta(A, B; \beta, \gamma) = q(-1) = \frac{1}{\beta} \left[\frac{\beta + \gamma}{F(1, \beta(\frac{B-A}{B}); \beta + \gamma + 1; \frac{-B}{1-B})} - \gamma \right].$$

(c) Furthermore, if $0 < B < 1$, $B < A \leq \min\{1 + \gamma(1-B)\beta^{-1}, (2\beta + \gamma + 1)B\beta^{-1}\}$, then

$$(6.17) \quad \delta(A, B; \beta, \gamma) = q(1) = \frac{1}{\beta} \left[\frac{\beta + \gamma}{F(1, \beta(\frac{A-B}{B}); \beta + \gamma + 1; \frac{B}{1+B})} - \gamma \right].$$

Under the hypotheses of Lemma 6.25, when $f \in \mathcal{S}^*(A, B)$, we get by [125, Theorem 2]

$$\|J_{\beta, \gamma}[f]\| \leq 6 - 4\delta,$$

where δ is given either by (6.16) or (6.17) with the corresponding conditions.

As a special case, we mention the following: if $f \in \mathcal{S}^*(\alpha)$ and β, γ are real numbers such that $\beta > 0$, $\beta + \gamma > 0$ and

$$\max \left\{ 0, -\frac{\gamma}{\beta}, \frac{\beta - \gamma - 1}{2\beta} \right\} \leq \alpha < 1,$$

then $J_{\beta, \gamma}[f]$ defined by (6.15) is in $\mathcal{S}^*(\delta)$, where

$$(6.18) \quad \delta = \delta(\alpha, \beta, \gamma) = \frac{1}{\beta} \left[\frac{\beta + \gamma}{F(1, 2\beta(1-\alpha); \beta + \gamma + 1; 1/2)} - \gamma \right].$$

Consequently, by [125, Theorem 2], we have the estimate

$$\|J_{\beta, \gamma}[f]\| \leq 6 - 4\delta,$$

where δ is given by (6.18).

In particular, for $f \in \mathcal{S}^*(\alpha)$ and $\max\{0, -\gamma\} \leq \alpha < 1$, we have $B_\gamma[f] \in \mathcal{S}^*(\delta(\alpha, \gamma))$, where

$$(6.19) \quad \delta = \delta(\alpha, \gamma) = \frac{\gamma + 1}{F(1, 2(1 - \alpha); \gamma + 2; 1/2)} - \gamma.$$

Thus, we have

$$\|B_\gamma[f]\| \leq 6 - 4\delta,$$

where δ is given by (6.19). Consequently, the following result gives a norm estimate for the Bernardi integral transform of functions that are not necessarily univalent.

Corollary 6.26. *Let $\gamma > -1$ and $f \in \mathcal{S}^*(-\gamma)$. Then*

$$\|B_\gamma[f]\| \leq 6 - 4 \left[\frac{\Gamma(\frac{3}{2} + \gamma)}{\sqrt{\pi} \Gamma(1 + \gamma)} - \gamma \right].$$

Proof. Recall the well-known identity (see [105, p. 69])

$$F(2a, 2b; a + b + 1/2; 1/2) = \frac{\Gamma(a + b + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(a + \frac{1}{2})\Gamma(b + \frac{1}{2})}.$$

Choose $a = 1/2, b = 1 - \alpha$ and $\alpha = -\gamma$. Then (6.19) yields

$$\delta(\gamma) = \delta(-\gamma, \gamma) = -\gamma + \frac{\Gamma(\frac{3}{2} + \gamma)}{\Gamma(1 + \gamma)\Gamma(\frac{1}{2})}$$

which may be written in terms of beta function given by

$$\delta(\gamma) = -\gamma + \frac{1}{B(1/2, 1 + \gamma)}.$$

Thus, for $f \in \mathcal{S}^*(-\gamma)$ we notice that $B_\gamma[f] \in \mathcal{S}^*(\delta(\gamma))$. Therefore, we have

$$\|B_\gamma[f]\| \leq 6 - 4\delta(\gamma)$$

and the conclusion follows. □

Problem 6.27. Find the sharp norm estimate for $B_\gamma[f]$ when $f \in \mathcal{S}^*(-\gamma)$. More generally, find a sharp norm estimate for $J_{\beta, \gamma}[f]$ whenever $f \in \mathcal{S}^*(\alpha)$, $\alpha < 1$.

A number of problems of this type may be raised for various integral transforms. For example, there exist conditions on $\lambda(t)$ and subfamilies \mathcal{F} of \mathcal{A} such that the integral transform of the form

$$V_\lambda[f](z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt \quad (f \in \mathcal{F})$$

is close-to-convex or starlike or convex, respectively (see [32, 98, 71] for details). In view of this, one can ask for the norm estimate for $V_\lambda[f]$ when f runs over suitable subclasses \mathcal{F} of \mathcal{A} . We remark that for the choice $\lambda(t) = (1 + \gamma)t^\gamma$ ($\gamma > -1$), $V_\lambda[f](z)$ reduces to the Bernardi transform of f .

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LIST OF PAPERS BASED ON THE THESIS

1. **P. Hästö, S. Ponnusamy and S.K. Sahoo** (2006) Inequalities and geometry of the Apollonian and related metrics, *Rev. Roumanie Math. Pure Appl.* **51**, no. 4, 433–452.
2. **S. Ponnusamy and S.K. Sahoo** (2006) Study of some subclasses of univalent functions and their radius properties, *Kodai Math. J.* **29**, no. 3, 391–405.
3. **P. Hästö, Z. Ibragimov, D. Minda, S. Ponnusamy and S.K. Sahoo** (2007) Isometries of some hyperbolic type path metrics and the hyperbolic medial axis, *In the Tradition of Ahlfors-Bers, IV (Ann Arbor, MI, 2005)*, 63–74, *Contemp. Math.* **432**, Amer. Math. Soc., Providence, RI.
4. **M. Huang, X. Wang, S. Ponnusamy and S.K. Sahoo** (2008) Uniform domains, John domains and quasi-isotropic domains, *J. Math. Anal. Appl.* **343**, 110–126.
5. **R. Parvatham, S. Ponnusamy and S.K. Sahoo** (2008) Norm estimates for the Bernardi integral transforms of functions defined by subordination, *Hiroshima Math. J.* **38**, 19–29.
6. **S. Ponnusamy and S.K. Sahoo** (2008) Norm estimates for convolution transforms of certain classes of analytic functions, *J. Math. Anal. Appl.* **342**, 171–180.
7. **P. Hästö, S. Ponnusamy and S.K. Sahoo**. Equivalence of the Apollonian and its inner metric, *In* Gustafsson, B. and Vasílev, A. (Eds.) *Analysis and Mathematical Physics* (Bergen, Norway, 2006) (submitted).
8. **S. Ponnusamy and S.K. Sahoo**. Pre-Schwarzian norm estimates of functions for a subclass of strongly starlike functions (submitted).

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